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P, S, *and* Q *refer to Proposals, Solutions and Quickies, respectively. Page numbers are given in parentheses. Thus,* P1175(239) *refers to proposal number* 1175 *which appears on page* 239.

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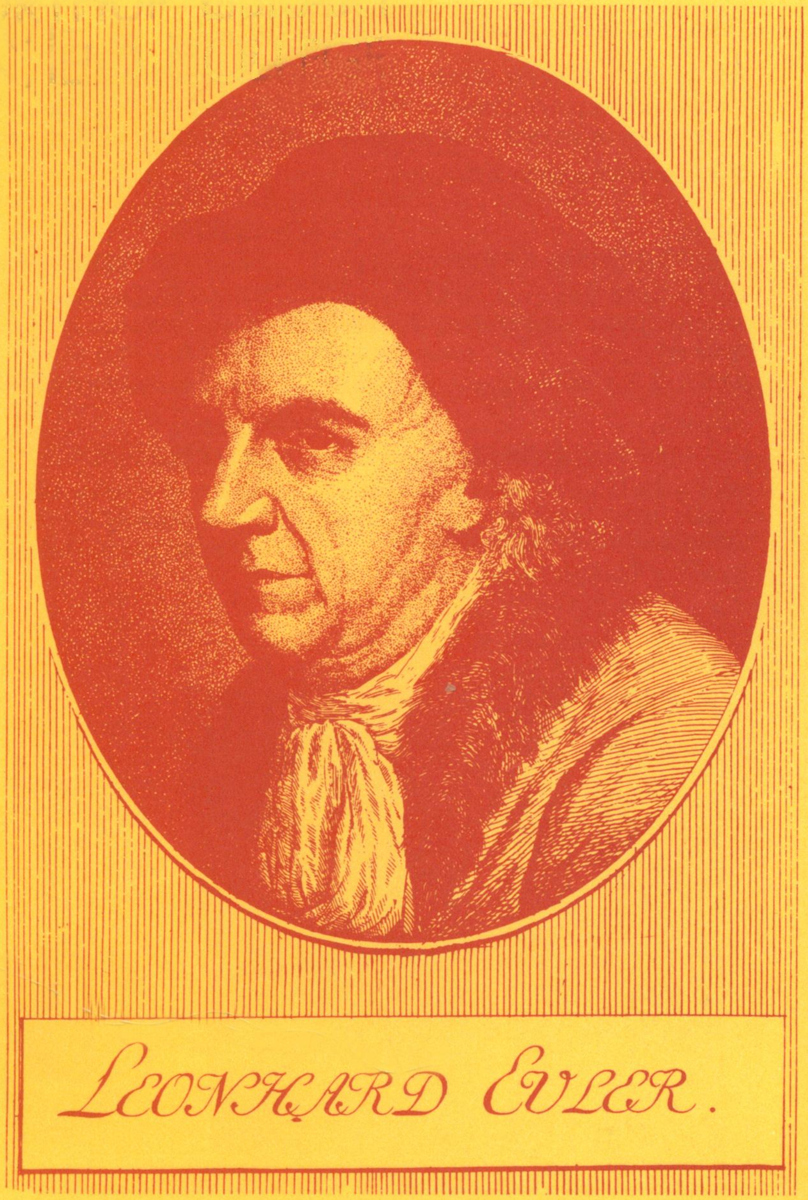
MATHEMATICS MAGAZINE



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A TRIBUTE TO LEONHARD EULER

1707–1783

MATHEMATICS MAGAZINE

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EDITORIAL POLICY

Mathematics Magazine is a journal which aims to provide inviting, informal mathematical exposition. Manuscripts accepted for publication in the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles submitted for publication should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

Authors planning to submit manuscripts should read the full statement of editorial policy which appears in this *Magazine*, Vol. 54, pp. 44–45, and is also available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

New manuscripts should be sent to: Doris Schattschneider, Editor, Mathematics Magazine, Moravian College, Bethlehem, PA 18018. Manuscripts should be prepared in a style consistent with the format of *Mathematics Magazine*. They should be typewritten and double spaced on $8\frac{1}{2}$ by 11 paper. Authors should submit the original and one copy and keep one copy as protection against possible loss. Illustrations should be carefully prepared on separate sheets of paper in black ink, the original without lettering and two copies with lettering added.

ABOUT THIS ISSUE

"In mathematics, the eighteenth century can fairly be labeled the age of Euler..." [**1**]. In his eulogy of Euler, M. de Condorcet observed, "All the noted mathematicians of the present day are his pupils: there is no one of them who has not formed himself by the study of his works, who has not received from him the formulas, the method which he employs; who is not directed and supported by the genius of Euler in his discoveries. This honour he owes to the revolution effected in the mathematical sciences, by subjecting all to analysis, to his indefatigable application, which has enabled him to embrace the whole extent of these sciences; to the order in which he has arranged his great works; to the simplicity, to the elegance, of his formulas; to the clearness of his methods and demonstrations; and all this greatly enhanced, by the multiplicity and the choice of his examples" [**2**].

Two hundred years after Euler's death (Sept. 17, 1783) the impact of his 'revolution' is very evident in all branches of mathematics. This special issue celebrates Euler's life and work—we want you to be awed, fascinated, and, finally, enticed to learn more. Reference [**1**] is a good place to start; our authors provide many others.

[1] A. P. Youschkevitch, Leonard Euler, Dictionary of Scientific Biography, v. IV, Charles Scribner's Sons, New York, 1971, pp. 467–484.

[2] Marquis de Condorcet, Eloge de M. Euler, in Histoire de l'Académie royale des sciences pour l'année 1783 (Paris, 1786), pp. 37–68.

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Subscription correspondence and notice of change of address should be sent to A. B. Willcox, Executive Director, Mathematical Association of America, 1529 Eighteenth Street, N. W., Washington, D.C. 20036. Back issues may be purchased, when in print, from P. and H. Bliss Co., Middletown, Connecticut 06457.

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AUTHORS

G. L. Alexanderson ("Ars Expositionis: Euler as Writer and Teacher") developed a special admiration for Euler when he was a student of G. Pólya at Stanford in the 1950's. In 1961 he bought a first edition of Euler's *Introductio in analysin infinitorum,* the start down the road to financial ruin, and he has since bought many old books by and about Euler, including the original editions of the *Proceedings of the St. Petersburg Academy.* From Pólya he also developed an interest in problems and is presently Associate Director of the William Lowell Putnam Mathematical Competition and a Problems Editor of the *American Mathematical Monthly.* Currently Michael and Elizabeth Valeriote Professor in Science at the University of Santa Clara, he has been chairman of the Department of Mathematics since 1967. He is co-author of several texts, the most recent *A First Undergraduate Course in Abstract Algebra,* 3rd edition (Wadsworth).

George Andrews ("Euler's Pentagonal Number Theorem") received his Ph.D. at the University of Pennsylvania in 1964. He was the late Professor Hans Rademacher's last student. Rademacher's lectures about Euler's work on partitions provided the inspiration for his Ph.D. thesis and much subsequent research. Currently Evan Pugh Professor of Mathematics at the Pennsylvania State University, he was a Fulbright scholar at Cambridge University in 1960–61, and has spent leaves of absence at M.I.T., Wisconsin and the University of New South Wales. He has published extensively on the theory of partitions and basic hypergeometric functions. His books are *Number Theory* (1971), *The Theory of Partitions* (1976), *Partitions: Yesterday and Today* (1979). He has recently edited *The Collected Papers of P. A. MacMahon.*

J. J. Burckhardt ("Leonhard Euler, 1707–1783") was born July 13, 1903, in Basel, Switzerland. He received his Ph.D. in 1927 from the University of Zurich, and taught mathematics at the University of Zurich from 1933–1977. He was Managing Editor of the journal *Commentarii Mathematici Helvetici* from 1950–1981, and served as president of the Swiss Mathematical Society 1954–56. He has several publications in the field of mathematical crystallography, as well as papers in the history of mathematics. He is a member of the editorial committee for the book *Euler-Gedenkband 1983.*

Underwood Dudley ("Some Remarks and Problems in Number Theory Related to the Work of Euler") showed early promise—he solved his first problem in the *American Mathematical Monthly* in 1954 and first wrote for this *Magazine* in 1960. Currently an associate editor of this *Magazine,* he also served in that capacity under editors L. Steen and J. A. Seebach. After being granted a Ph.D. from the University of Michigan (under W. J. LeVeque), he taught briefly at the Ohio State University before joining the faculty of DePauw University, at which he expects to stay until the pressure for early retirement becomes irresistible. He has written several papers on number theory and the history of mathematics. He is an avid collector of angle trisections, proofs of Fermat's Last Theorem, and other material supplied by undaunted correspondents.

Harold M. Edwards ("Euler and Quadratic Reciprocity") has published three books—*Advanced Calculus, Riemann's Zeta Function,* and *Fermat's Last Theorem*—all containing admiring references to Euler's work (for example, see pages 426, 12, and 286, respectively). He received a B.A. from the University of Wisconsin in 1956 and a Ph.D. from Harvard in 1961, and has been on the faculties of Harvard, Columbia, and New York University. He is now Professor of Mathematics at NYU. His fourth book, *Galois Theory,* will be published this year.

Paul Erdős ("Some Remarks and Problems in Number Theory Related to the Work of Euler") was born on March 26, 1913. "Both of my parents were mathematicians, and I learned a great deal from them. When I was 4 years old I told my mother that if you take away 250 from 100 you get 150 below 0. I learned from my father at 10 that the number of primes is infinite but that there are arbitrarily large gaps between the primes. One of my main interests always remained number theory; the second was combinatorics and set theory. At one time I learned from an old book of Euler, *Anleitung zur Algebra,* how to solve cubic and quartic equations and that $x^4+y^4=z^2$ has no nontrivial solution. Thus my interest in Euler started early."

After receiving his Ph.D. at the University of Budapest in 1934, Erdős went immediately to Manchester, England. Subsequently he visited the U.S., Holland, and Hungary, and spent 1949–54 in the U.S. and England. In 1954 he visited Holland, Switzerland and Israel, returning to Hungary again in 1955. "Since then I constantly travel around the world, rarely staying in the same place for more than a month (there is plenty of time to settle down in the grave). I have always worked easily and well with other mathematicians and have more than 200 co-authors."

Morris Kline ("Euler and Infinite Series") has served primarily at New York University where he was founder and Director of the Division of Electromagnetic Research of the Courant Institute and Chairman of the undergraduate mathematics program at the Washington Square center. He has been a visiting professor at Stanford University, the Technische Hochschule in Aachen, Germany, and a Distinuished Visiting Professor at Brooklyn College of the City University of New York. His awards include a Guggenheim Fellowship, Fulbright Lecturer in Germany, and a Great Teacher Award at New York University. He is the author of several books, the latest of which is *Mathematics, The Loss of Certainty* (Oxford University Press, 1980). Beyond research, his interests and writing have been in history and pedagogy.

Jesper Lützen ("Euler's Vision of a Generalized Partial Differential Calculus for a Generalized Kind of Function") became interested in Euler's ideas on functions and their application in analysis when he composed his thesis in 1976 on the history of the function concept for the master's degree in mathematics and physics at Aarhus University, Denmark. He later developed some aspects of it in his 1980 Ph.D. thesis on the prehistory of the theory of distributions. Parts of this thesis were written during a stay at Yale University. After a temporary position in the Department of History of Science at Aarhus University, he currently holds a research scholarship in the Mathematics Department at Odense University which allows him to devote his time to a study of the life and works of the French mathematician Joseph Liouville.

ILLUSTRATIONS

The editor expresses deep appreciation to the many people whose generous cooperation made possible the illustrations that appear in this issue.

Lehigh University Library, special collections, made available to the editor their original Euler works and early translations. Georgia Raynor was especially helpful in assisting the editor in the task of photographing from these volumes. Photos from the Lehigh collection appear on pp. 266, 275 (left), 299, 300, 302, 313.

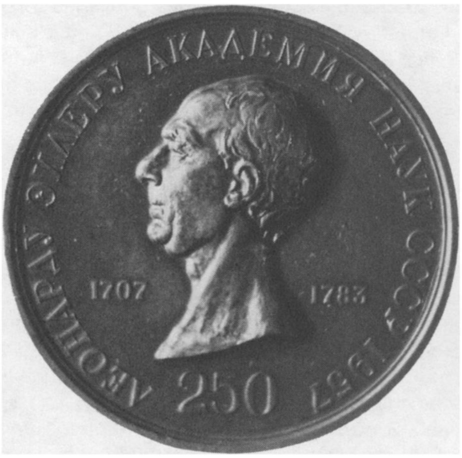
Mrs. Albert DeNeufville provided the editor with original Euler works from her husband's collection (see pp. 275 [right], 294–5).

Photocopies of pages from Euler's original publications were provided by J. J. Burckhardt and G. Alexanderson.

All portraits of Euler that appear in this issue were provided by J. J. Burckhardt, on behalf of the editorial committee for the 1983 Euler memorial volume, commissioned by the Canton of Basel, Switzerland.

The Swiss banknote and commemorative medal of Euler were provided by Rodney T. Hood.

Cover: Stipple engraving of Leonhard Euler by R. Blake, based on the plaster relief by J. Rachette, 1781; Academy of Sciences, Paris. Photographed by Marcel Jenni, University Library, Basel.



Commemorative bronze medallion (6 cm. diameter), struck by the Academies of Science of Berlin and of the USSR in 1957 to celebrate the 250th anniversary of Euler's birth, shows a relief of Euler sculpted by G. S. Shklovsky, after a bust of Euler by J. Rachette.

ACKNOWLEDGMENTS

In addition to our associate editors, the following have assisted the Magazine *by refereeing papers during the past year. We appreciate the time they have spent and care they have shown in this task.*

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Olinick, Michael, *Middlebury College.*
Porter, Gerald J., *University of Pennsylvania.*
Ridge, Jack, *Moravian College.*
Rigby, John F., *University College, Cardiff.*
Rosen, David, *Swarthmore College.*

ARTICLES

Leonhard Euler, 1707–1783

J. J. BURCKHARDT
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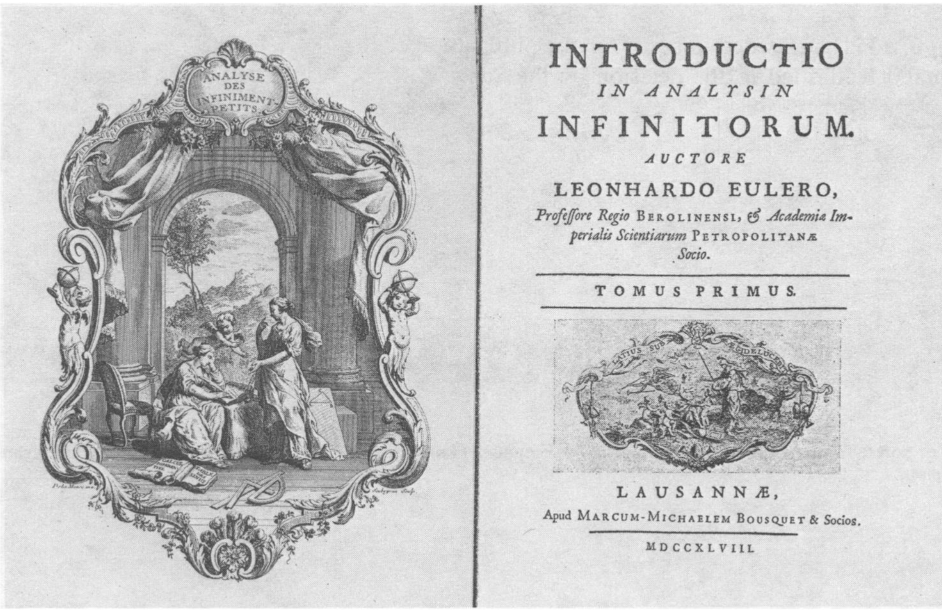
Switzerland

Born in 1707, Leonhard Euler grew up in the town of Riehen, near Basel, Switzerland. Encouraged by his father, Paulus, a minister, young Leonhard received very early instruction from Johann I Bernoulli, who immediately recognized Euler's talents. Euler completed his work at the University of Basel at age 15, and at age 19 won a prize in the competition organized by the Academy of Sciences in Paris. His paper discussed the optimal arrangement of masts on sailing ships (*Meditationes super problemate nautico...*). In 1727 Euler attempted unsuccessfully to obtain a professorship of physics in Basel by submitting a dissertation on sound (*Dissertatio physica de sono*); however, this failure, in retrospect, was fortunate. Encouraged by Nicholas and Daniel, sons of his teacher Johann Bernoulli, he went to the St. Petersburg Academy in Russia, a field of action that could accommodate his genius and energy.

In St. Petersburg Euler was met by compatriots Jacob Hermann and Daniel Bernoulli and soon befriended the diplomat and amateur mathematician Christian Goldbach. During the years 1727–1741 spent there, Euler wrote over 100 scientific papers and his fundamental work on mechanics. In 1741, at the invitation of Fredrick the Great, he went to the Akademie in Berlin. During his 25 years in Berlin, his incredible mathematical productivity continued. He created, among other works, the calculus of variations, wrote the *Introductio in analysin infinitorum*, and translated and rewrote the treatise on artillery by Benjamin Robins.



INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLVIII.

Pastel portrait of Leonhard Euler painted by Emanuel Handmann, 1753; Kunstmuseum, Basel. Photographed by August Hinz, Basel.

Disputes with the Court led Euler in 1766 to accept a very favorable invitation by Katherine II to return to St. Petersburg. There he was received in a princely manner, and he spent the rest of his life in St. Petersburg. Although totally blind, he wrote, with the help of his students, the famous *Algebra* and over 400 scientific papers; he left many unpublished manuscripts.

In recent decades, numerous important materials concerning Euler have been discovered in the archives in the Academy of Sciences of the USSR. It would seem that there is probably little chance of now discovering an unknown manuscript or something important about his life. Euler himself acknowledged the advantageous circumstances he found at the Academy. Judith Kh. Kopelevič notes, "Euler's tombstone, erected by the Academy; his bust in the building of the Presidium of the Academy; the two-centuries-long efforts of the Academy to care for his enormous heritage and publish it—all these show clearly that Euler's encounter with the Petersburg Academy of Sciences was a happy one for both sides."

The legacy of Euler's writings

Euler's productivity is astonishing in its range of content and in the sheer volume of written pages. He wrote landmark books on the subjects of mathematical analysis, analytic and differential geometry, the calculus of variations, mechanics, and algebra. He published over 760 research papers, many of which won awards in competitions, and at his death left hundreds of unpublished works; even today there remain unpublished over 3,000 pages in notebooks. In view of this prodigious collection of written material, it is not surprising that soon after Euler's death the task of surveying and publishing his works encountered extraordinary difficulty.

N. I. Fuss made efforts to publish more writings of the master, but only his son P.-H. Fuss succeeded (with the help of C. G. J. Jacobi) to generate interest among others, including Ostrogradskii. An enterprise in this direction was undertaken in Belgium (1838–1839), but failed after the publication of the fifth volume. In 1844, the Petersburg Academy decided on publication of the manuscripts, but this was not carried out. However, in 1849 the *Commentationes arithmeticae collectae*, edited by P.-H. and N. Fuss, were published; this contains, among others, the important manuscript *Tractatus de doctrina numerorum*.

The centennial of Euler's death in 1883 rekindled interest in Euler's works and in 1896 the most valuable preliminary to any complete publication appeared—the *Index operum Leonhardi Euleri* by J. G. Hagen. As the bicentennial of Euler's birth neared, new life was infused into the project, which was thoroughly discussed by the academies of Petersburg and Berlin in 1903. Although the project was abandoned at this time, the celebrations of the bicentennial of Euler's birth provided the needed impetus for the publication of the *Opera omnia*. The untiring efforts of Ferdinand Rudio led to the decision by the Schweizerische Naturforschende Gesellschaft [Swiss Academy of Sciences] in 1909 to undertake the publication, based on the list of Euler's writings prepared by Gustaf Eneström (1910–1913). He lists 866 papers and books published by then. The financial side appeared assured through gifts and subscriptions. But the first World War led to unforeseen difficulties. We are indebted to Andreas Speiser for his efforts, which made it possible to continue the publication, and who overcame financial and publication difficulties so that at the start of World War II about one half of the project was completed. After the war, Speiser, succeeded by Walter Habicht, completed the series 1 (29 volumes), 2 (31 volumes) and 3 (12 volumes) of the *Opera omnia* except for a few volumes.

In 1947–1948 the manuscripts which had been loaned by the St. Petersburg Academy to the Swiss Academy of Sciences were returned to the archives of the Academy of Sciences of the USSR in Leningrad. Their systematic study was started under the supervision of the Academician V. I. Smirnov, with the goal of publishing a fourth series of the *Opera omnia*. As a first result, there appeared in 1965 a new edition of the correspondence between Euler and Goldbach, edited by A. P. Juškevič and E. Winter. In 1967, the Swiss Academy of Sciences and the Academia Nauk of the USSR formed an International Committee, to which was entrusted the publication of Euler's correspondence in a series 4A, and a critical publication of the remaining manuscripts in a series 4B.

To mark the passage of 200 years since Euler's death, a memorial volume has been produced by the Canton of Basel, *Leonhard Euler 1707–1783, Beiträge zu Leben und Werk*, edited by J. J. Burckhardt, E. A. Fellmann, and W. Habicht (Birkhäuser Verlag, Basel and Boston). From a contemporary point of view, this volume presents the insights of outstanding scientists on various aspects of Euler's achievements and their influence on later works. The complete list of essays and their authors appears at the end of this article. The memorial volume ends with a list, compiled by J. J. Burckhardt, of over 700 papers which are devoted to the work of Euler. It should be stressed that this is certainly an incomplete list, and it is hoped that it will lead to many additional listings which will then be published in an appropriate form. It is hoped that papers little known till now will receive the attention they deserve, and that this effort will lead to an improvement in the collaboration of scientists of all countries.

In the present article, we give a brief overview of the work of Euler. In order to include information from recently discovered work as well as the observations and insights of modern scholars, we draw freely from material found in the memorial volume.

Number Theory

Euler had a passionate lifelong interest in the theory of numbers. Approximately one-sixth of his published work in pure mathematics is in this area; the same is true of the manuscripts left unpublished at his death. Although he had an active correspondence with Goldbach, he complained about the lack of response on the part of other contemporary mathematicians such as Huygens, Clairaut, and Daniel Bernoulli, who considered number theory investigations a waste of time, and were even unaware of Fermat's Theorem. (Forty years passed before Euler's investigations into Goldbach's problem were followed up by Lagrange.)

André Weil has commented that if one were to distinguish between "theoretical" and "experimental" researchers, as is done for physicists, then Euler's constant preoccupation with number *theory* would place him among the former. But in view of his insistence on the "inductive" method of discovery of arithmetic truths, carrying out a wealth of numerical calculations for special cases before tackling the general question, one could equally well call him an "experimental" genius.

At the beginning of the eighteenth century—50 years after Fermat's death—the number theoretical work of Fermat was practically forgotten. In a letter dated December 1, 1727, Christian Goldbach brought to Euler's attention Fermat's assertion that numbers of the form

$$2^{2^{p-1}}+1,\ p \text{ prime},$$

(i.e., $3, 5, 17, 257, \dots$) are also prime; this led Euler to a study of Fermat's works. His investigations included Fermat's Theorem and its generalizations, representations of numbers as sums of squares of polygonal numbers, and elementary quadratic forms.

In the decade between 1740 and 1750, Euler created the basis of a new theory which, until this day, has not essentially changed its character. The question which motivated this work was posed by Naudé on September 12, 1740, who asked Euler the number of ways in which a given integer can be represented as a sum of integers. For this problem, the "partitio numerorum," as well as for related problems, Euler found solutions by associating with a number-theoretic function its generating function, which can be investigated by analytical methods. Euler clearly understood the importance of his discovery. Although he had not found the proof of several central theorems of his theory, he incorporated the basic ideas and a few elementary but remarkable special results in his fundamental text in analysis, *Introductio in analysin infinitorum*. V. Scharlau comments, "Even today it is hard to imagine a more convincing and interesting introduction to this theory."

Euler used this theory in attempting to find a formula for prime numbers, where he considered the function $\sigma(n)$, the sum of all divisors of n. He obtained the formula

$$\sigma(p^k)=(p^{k+1}-1)/(p-1), \text{ for } p \text{ prime}$$

from which the computation of $\sigma(n)$ follows. Euler also formulated the recursion rule for $\sigma(n)$,

$$\sigma(n)=\sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\cdots$$

and observed its similarity to the one for $p(n)$, the number of partitions of n. In 1750, Euler brought these investigations to a conclusion by formulating the identity

$$\prod_{i=1}^{\infty}(1-x^{i})=1+\sum_{m=1}^{\infty}(-1)^{m}\left(x^{\frac{1}{2}(3m^{2}-m)}+x^{\frac{1}{2}(3m^{2}+m)}\right)$$

which is a cornerstone for all his related results.

Another interesting application of generating functions can be found in Euler's various investigations of "population dynamics," which probably originated in the years 1750–1755. Scharlau writes:

> From today's point of view it is possibly not surprising that Euler found no additional results on generating functions; indeed it took many decades—almost a century—after the end of his activity before his achievements were substantially surpassed. It is remarkable how little attention was given to Euler's ideas by the mathematicians of the 18th and 19th centuries.... There are very few mathematical theories whose character has changed so little since Euler's time as the theory of generating functions and the partitions of numbers.

Among the unpublished fragments of Euler's work (a total of about 3,000 pages, mainly bound in numbered notebooks) are over 1,000 pages which are devoted to number theory, mostly from the years 1736–1744 and 1767–1783. Euler's technique of investigation emerges clearly from these. After lengthy efforts which at times span many years, he reaches his results based on observations, tables, and empirically established facts.

G. P. Matvievskaja and E. P. Ozigova, who have perused these fragments, note that "the handwritten materials widen our views of Euler's activity in the field of number theory. The same holds for other directions of his research. The manuscripts enable us to recognize the sources of his mathematical discoveries." A few examples serve to illustrate these points. On page 18 in notebook N 131 is the problem of deciding whether a given integer is prime. The same notebook contains an entry about the origin of the zeta function, as well as the first mention of the theorem of four squares, to which Euler returns in notebook N 132 (1740–1744). A particularly interesting entry in notebook N 134 (1752–1755) contains Euler's formulation, a hundred years before Bertrand, of the "Bertrand postulate," that there is at least one prime between any integer n and $2n$.

Analysis

Euler was occupied throughout his life with the concept of function; the treatises he produced in analysis were fundamental to the development of the modern foundations of analysis. As early as 1727 Euler had written a fifteen-page manuscript *Calculus differentialis*; it's interesting to compare this fledgling work with his later treatise *Institutiones calculi differentialis* (1755). Here Euler explains the calculus of finite differences of finite increments and considers calculations with infinitely small quantities. D. Laugwitz, one of the contributors to the modern development of analysis through the adjoining of an infinity symbol Ω, remarks that anyone who reads this work, or Euler's *Introductio in analysin infinitorum* (1748), must be struck by the confidence with which Euler utilizes the calculus of both infinitely large and infinitely small magnitudes. Laugwitz indicates that it is possible to formulate Euler's ideas in the modern setting of nonstandard analysis, hence Euler receives a belated justification of his unorthodox techniques.

The richness and diversity of Euler's work in analysis can be seen by a brief summary of the book *Introductio in analysin infinitorum*. The first chapter discusses the definition of "function" which originated with Johann Bernoulli. In the second, Euler formulates the "fundamental theorem of algebra" and sketches a proof; he presents results on real and complex solutions of algebraic equations, a topic resumed in chapter 12 which deals with the decomposition of rational functions into partial fractions. The third chapter contains the so-called "Euler substitution," and

EVOLUTIONE FACTORUM ORTIS. 225

ubi alii numeri non occurrunt, niſi qui ex his duobus 2 & 3 per multiplicationem originem trahunt; ſeu qui alios Diviſores præter 2 & 3 non habent. CAP. XV.

273. Si igitur pro $\alpha, \beta, \gamma, \delta$, &c., unitas per ſingulos omnes numeros primos ſcribatur, ac ponatur

$$P = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})\ \&c.,}$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi omnes numeri tam primi, quam qui ex primis per multiplicationem naſcuntur, occurrunt. Cum autem omnes numeri vel ſint ipſi primi, vel ex primis per multiplicationem oriundi, manifeſtum eſt, hic omnes omnino numeros integros in denominatoribus adeſſe debere.

274. Idem evenit, ſi numerorum primorum Poteſtates quæcunque accipiantur: ſi enim ponatur

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.,}$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod ſi autem in Factoribus ubique ſignum + ſtatuatur, ut ſit

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.,}$$

erit

Euleri *Introduct. in Anal. infin. parv.* Ff $P =$

DE PARTITIONE NUMERORUM. 253

CAP. XVI.

CAPUT XVI.

De Partitione numerorum.

297. Propoſita ſit iſta expreſſio

$$(1 + x^{\alpha}z)(1 + x^{\beta}z)(1 + x^{\gamma}z)(1 + x^{\delta}z)(1 + x^{\epsilon}z)\ \&c.,$$

quæ cujuſmodi induat formam, ſi per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

atque manifeſtum eſt P fore ſummam Poteſtatum $x^{\alpha} + x^{\beta} + x^{\gamma} + x^{\delta} + x^{\epsilon} + \&c.$. Deinde Q eſt ſumma Factorum ex binis Poteſtatibus diverſis, ſeu Q erit aggregatum plurium Poteſtatum ipſius x, quarum Exponentes ſunt ſummæ duorum terminorum diverſorum hujus Seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c.$$

Simili modo R erit aggregatum Poteſtatum ipſius x, quarum Exponentes ſunt ſummæ trium terminorum diverſorum. Atque S erit aggregatum Poteſtatum ipſius x, quarum Exponentes ſunt ſummæ quatuor terminorum diverſorum ejuſdem Seriei, $\alpha, \beta, \gamma, \delta, \epsilon$, &c., & ita porro.

298. Singulæ hæ Poteſtates ipſius x, quæ in valoribus literarum P, Q, R, S, &c., inſunt, unitatem pro coëfficiente habebunt, ſiquidem earum Exponentes unico modo ex

Ii 3 $\alpha, \beta,$

the important replacement of a non-explicit functional dependence by a parametric representation. Particularly remarkable is Euler's strict theory of logarithms, and the consideration of the exponential function in chapter 6. Euler asserts that the logarithms of rationals are either rational or transcendental, a fact which was proved only two hundred years later. Weakly convergent series are considered in chapter 7, as well as the question of convergence of series and the relation between a function and its representation outside the circle of convergence. Subsequent chapters deal with transcendental functions and their representation as series or products. The starting point of Bernhard Riemann's investigation of the distribution of primes is in chapter 15, in the formula

$$\sum_{n} \frac{1}{n^x} = \prod_{p} \left(\frac{1}{1 - 1/p^x} \right)$$

in which the summation extends over all positive integers and the product over all primes (see above, left). In chapter 16 Euler turns to the new topic—rife with algebraic ideas—of *Partitione numerorum*, the additive decomposition of natural numbers (see above, right). The developments of power series into infinite series found here were continued only by Ramanujan, Hardy and Littlewood. The expressions found here were later called theta functions, and used by Jacobi in the general theory of elliptic functions. The last chapter, 17, deals with the numerical solution of algebraic equations, following Daniel Bernoulli.

A. O. Gelfond, whose essay in the memorial volume contains a deep analysis of the contents of *Introductio...*, interprets Euler's ideas in modern terms and stresses the great relevance of this work, even to this day.

Euler's interest in the theory of vibrating strings is legendary. In 1747 d'Alembert formulated the theory and the corresponding partial differential equation; this prompted Euler in 1750 to develop a solution, although restricted to the case in which the vibrations satisfy certain conditions. Euler's friend Daniel Bernoulli contributed (about 1753) two remarkable articles, and presented the solution in the form of a trigonometric series. The problem is fittingly illuminated by Euler's question "what is the law of the vibrating string if it starts with an arbitrary shape" and d'Alembert's answer "in several cases it is not possible to solve the problem, which transcends the resources of the analysis available at this time."

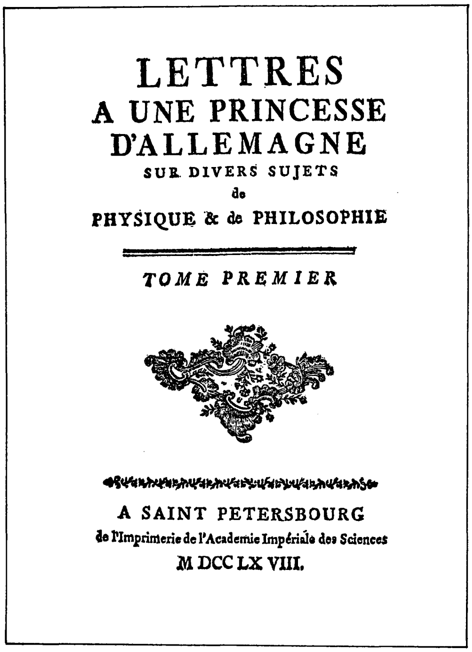
Euler has sometimes been criticized for seeming to ignore the concept of convergence in his freewheeling calculations. Yet in 1740, Euler gave an incomplete formulation of the criterion of convergence that later received Cauchy's name. Euler's last paper was completed in 1783, the year of his death; it contained the germ of the concept of uniform convergence. His example was utilized by Abel in 1826.

After surveying the rich contributions to analysis made in Euler's time, Pierre Dugac declares, "Euler and d'Alembert were the instigators of the most important work on the foundations of analysis in the nineteenth century."

"Applied" Mathematics (Physics)

Euler's investigations and formulations of basic theory in the areas of optics, electricity and magnetism, mechanics, hydrodynamics and hydraulics are among the most fundamental contributions to the development of physics as we know it today. Euler's views on physics had an immediate influence on the study of physics in Russia; this grew out of his close relationship with the contemporary and most influential Russian scientist, M. V. Lomonosov, his several Russian students, and the publication of a translation (by S. J. Rumovskii) of his very popular "Letters to a German Princess." The "Letters...," which had originated as lessons to the princess of Anhalt-Dessau, niece of the King of Prussia, during Euler's years in Berlin, served as the first encyclopedia of physics in Russia. A. T. Grigor'jan and V. S. Kirsanov have noted that the physicist N. M. Speranskii, a noted statesman and author of a physics book (1797), used to read to his students sections from Euler's "Letters...."

B. L. van der Waerden, in discussing Euler's justification of the principles of mechanics, has asked, "What did Euler mean by saying that in the computation of the total moment of all forces, the inner forces can be neglected because 'les forces internes se détruisent mutuellement'?" He points out that in order to answer that question it is important to know Euler's concept of solids, fluids, and gases. Are they true continua, or aggregates of small particles? The answer can be found in Euler's letters #69 and #70 to a German princess. He does not consider water, wool and air as true continua, but assumes that they consist of separate particles. However, in hydrodynamics, Euler treats liquids and gases as if they were continua. Euler is well aware that this is only an approximation.


LETTRES
A UNE PRINCESSE
D'ALLEMAGNE
SUR DIVERS SUJETS
de
PHYSIQUE & de PHILOSOPHIE

TOME PREMIER

A SAINT PETERSBOURG
de l'Imprimerie de l'Academie Impériale des Sciences
M DCC LX VIII.

A study of the published works of Daniel and Johann Bernoulli, as well as Euler's unpublished works (in particular, Euler's thick notebook from 1725–1727), by G. K. Mikhailov, gives some new and surprising insights into Euler's contributions to the development of theoretical hydraulics. Mikhailov states:

> It is generally known that the creation of the foundations of modern hydrodynamics of ideal fluids is one of the fruits of Euler's scientific activity. Less well known is his role in the development of theoretical hydraulics, that is, as usually understood, the hydrodynamic theory of fluid motion under a one-dimensional flow model. Traditionally—and with good reason—it is assumed that the foundations of hydraulics were developed by Daniel and Johann Bernoulli in their works published between 1729 and 1743. In fact, during the second quarter of the eighteenth century Euler did not publish even a single paper on the elements of hydraulics. The central theme of most of the recent historical-critical studies on the state of hydraulics in that period is the determination of the respective contributions of Daniel and of Johann Bernoulli. But Euler stood, all this time, just beyond the curtain of the stage on which the action was taking place, although almost no contemporary was aware of that.

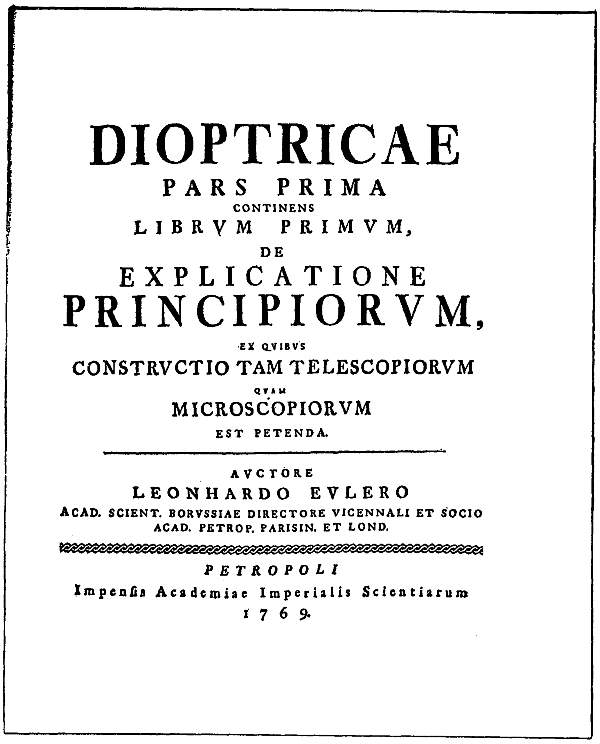
Euler's work on the theory of ships culminated in the publication of *Scientia navalis seu tractatus de construendis ac dirigendis navibus*, published in 1749. Walter Habicht notes the fundamental importance of this treatise:

> Following the *Mechanica sive motus scientia analytice exposita* which appeared in 1736, it [the *Scientia navalis*...] is the second milestone in the development of rational mechanics, and to this day has lost none of its importance. The principles of hydrostatics are presented here, for the first time, in complete clarity; based on them is a scientific foundation of the theory of shipbuilding. In fact, the topics treated here permit insights into all the related developments in mechanics during the eighteenth century.

Although Euler's intense interest in the science of optics appeared before he was 30 and remained with him almost to his death, there is still no monographic evaluation of his contributions to the wide field of physical and geometrical optics. Part of Euler's work is best described by Habicht:

> In the second half of his life, from 1750 on and throughout the sixties, Leonhard Euler worked intensively on problems in geometric optics. His goal was to improve in several ways optical instruments, in particular, telescopes and microscopes. Besides the determination of the enlargement, the light intensity and the field of view, he was primarily interested in the deviations from the point-by-point imaging of objects (caused by the diffraction of light passing through a system of lenses), and also in the even less tractable deviations which arise from the spherical shape of the lenses. To these problems Euler devoted a long series of papers, mainly published by the Berlin academy. He admitted that the computational solution of these problems is very hard. As was his custom, he collected his results in a grandly conceived textbook, the *Dioptrica* (1769–1771). This book deals with the determination of the path of a ray of light through a system of diffracting spherical surfaces, all of which have their centers on a line, the optical axis of the system. In a first approximation, Euler obtains the familiar formulae of elementary optics. In a second approximation he takes into account the spherical and chromatic aberrations. After passing through a diffracting surface, a pencil of rays issuing from a point on the optical axis is spread out in an interval on the optical axis; this is the so-called "longitudinal aberration." Euler uses the expression "espace de diffusion." If the light passes through several diffracting surfaces, the "espace de diffusion" is determined using a principle of superposition.
>
> Euler had great expectations for his theory, and believed that using his recipes, the optical instruments could be brought to "the highest degree of perfection." Unfortunately, the practical realization of his systems of lenses did not yield the hoped-for success. He searched for the causes of failure in the poor quality of the lenses on the one hand, and also in basic errors in the laws of diffraction which were determined experimentally in a manner completely unsatisfactory from a theoretical point of view. Because of the failure of his predictions, Euler's *Dioptrica* is often underrated.

Habicht notes that Euler's theory can be modified to obtain the general imaging theories developed in the nineteenth century. The crucial gap in Euler's treatment consists in neglecting



DIOPTRICAE
PARS PRIMA
CONTINENS
LIBRVM PRIMVM,
DE
EXPLICATIONE
PRINCIPIORVM,
EX QVIBVS
CONSTRVCTIO TAM TELESCOPIORVM
QVAM
MICROSCOPIORVM
EST PETENDA.

AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LOND.

PETROPOLI
Impensis Academiae Imperialis Scientiarum
1769.

those aberrations which are caused by the distance of the object and its images from the optical axis; with modification it is possible to determine the spherical aberration errors of the third order directly from Euler's formulas.

A responsible evaluation of Euler's contributions to optics will be possible only after Euler's unpublished letters and manuscripts are edited and made generally accessible. E. A. Fellmann provides an example of Euler's method which helps to place Euler's contribution in a historic context. The problem of diffraction in the atmosphere is one which was first seriously considered by Euler:

> He began by deriving a very general differential equation; naturally, it turned out not to be integrable—it would have been a miracle had that not happened. Then he searched for conditions which make a solution possible, and finally he solved the problem in several cases under practically plausible assumptions.
>
> Euler frequently expressed the opinion that the phenomena in optics, electricity and magnetism are closely related (as states of the ether), and that therefore they should receive simultaneous and equal treatment. This prophetic dream of Euler concerning the unity of physics could only be realized after the construction of bridges (experimental as well as theoretical) which were missing in Euler's time. These were later built by Faraday, W. Weber and Maxwell.

Euler was deeply influenced by the work of scientists who preceded him as well as by the work of his contemporaries. This is perhaps best illustrated by his role in the development of potential theory. He acknowledges the influence of the work of Leibniz, the Bernoullis, and Jacob Hermann, whose work he had studied in his days in Basel to 1727. In the decade 1730–1740, the contemporaries Euler, Clairaut and Fontaine all were active in developing the main ideas that would lead to potential theory: the geometry of curves, the calculus of variations, and the study of mechanics. By 1752 Euler's work on fluid mechanics *Principia motus fluidorum* was complete. A summary of his contributions to potential theory is given by Jim Cross:

He helped, with Fontaine and Clairaut, to develop a logical, well-founded calculus of several variables in a clear notation; he transformed, with Daniel Bernoulli and Clairaut, the Galileo-Leibniz energy equation for a particle falling under gravity, into a general principle applicable to continuous bodies and general forces (the principle of least action with Daniel Bernoulli and Maupertuis forms part of this); and he founded, after the attempts of the Bernoullis, d'Alembert, and especially Clairaut, the modern theory of fluid mechanics on complete differentials for forces and velocities. His work was fruitful: the theories of Lagrange grew from his writings on extremization, fluids and sound, and mechanics; the work of Laplace followed.

Astronomy

Research by Nina I. Nevskaja based on newly available original documents justifies calling Euler a professional astronomer—and even an observer and experimental scientist. Five hundred books and manuscripts from the private library of Joseph Nicholas Delisle have recently come to light and from these one finds that this scientist found Euler a suitable collaborator and valued his knowledge in spherical trigonometry, analysis and probability.

It was a surprise when the records of observations of the Petersburg observatory during its first 21 years—which were presumed lost—were discovered in 1977 in the Leningrad branch of the archives of the Academy of Sciences of the USSR. For almost ten years, Euler was among those who were regularly taking measurements twice daily. Based on these observations, Delisle and Euler computed the instant of true noon, and the noon correction. Euler's entries were so detailed and numerous that it is possible to deduce from them how he gradually mastered the methods of astronomical observations. Utilizing the insights he obtained, Euler found a simple method of computing tables for the meridional equation of the sun; he presented it in the paper *Methodus computandi aequationem meridiei* (1735).

Euler was fascinated by sunspots; his notes from this period contain enthusiastic comments on his observations. The computation of the trajectories of the sunspots by Delisle's method can be considered the beginning of celestial mechanics. The archives also disclose that Euler helped Delisle by working out analytical methods for the determination of the paths of comets.

A little-noted field of Euler's activities, the theory of motion of celestial bodies, is documented by Otto Volk. Euler's first paper, based on generally formulated differential equations of mechanics, is entitled *Recherches sur le mouvement des corps célestes en general* (1747). Using the tables of planets computed by Thomas Street from the pure Keplerian motion of planets around the sun, Euler discusses in Sections 1 to 17 the observed irregularities. In Section 18 he formulates the differential equations of mechanics, and obtains the solution

$$r = a(1 + e\cos v) = \frac{a(1-e^2)}{1 - e\cos\phi}$$

in which r is the radius, v is the eccentric anomaly and ϕ is the true anomaly, while e and a are constants. This is a regularization of the so-called inverse problem of Newton. Later, Euler obtains a trigonometric series for ϕ; such Fourier series are the basis of his computation of perturbations. This is the topic treated in detail in the prize proposal to the Paris Academy, *Recherches sur la question des inégalités du mouvement de Saturne et de Jupiter, sujet proposé pour le prix de l'année 1748*. In it Euler uses, for the first time, Newton's laws of gravitation to compute the mutual perturbations of planets.

In his paper *Considerationes de motu corporum coelestium* (1764), Euler is the first to begin considering the three-body problem, under certain restrictions. Euler notes the intractability of the problem:

> There is no doubt that Kepler discovered the laws according to which celestial bodies move in their paths, and that Newton proved them—to the greatest advantage of astronomy. But this does not mean that the astronomical theory is at the highest level of perfection. We are able to deal completely with Newton's inverse-square law for two bodies. But if a third body is involved, so that each attracts both other bodies, all the arts of analysis are insufficient.... Since the solution of the general problem of three bodies appears to be beyond the human

powers of the author, he tried to solve the restricted problem in which the mass of the third body is negligible compared to the other two. Possibly, starting from special cases, the road to the solution of the general problem may be found. But even in the case of the restricted problem the solution encounters difficulties so great that the author has to admit to have spent much effort in vain attempts at solution.

Euler's investigation of the three-bodies problem was noted only at a later date; the linear solutions to the equation of the fifth degree were (and sometimes still are) called "Lagrange's solutions," without any mention of Euler. But Euler achieved fame through his theory of perturbations, presented in *Nouvelle méthode de déterminer les dérangemens dans le mouvement des corps célestes, causé par leur action mutuelle*. By iteration he determined, for the first time, the perturbations of the elements of the elliptical paths, and then applied this method to determine the motion of three mutually attracting bodies.

Correspondence

The circle of contemporary scholars who were influenced by and in turn, influenced, Euler's investigations was as wide as one could imagine in the eighteenth century. His voluminous correspondence testifies to the fruitful interaction between scientists through queries, conjectures, critical comments, and praise. Some of the correspondence has been published previously in collected works; a standard reference is the collection *Correspondance Mathématique et Physique*, edited by N. Fuss and published in 1843 by the Imperial Academy of Science, St. Petersburg. New discoveries and more complete information have produced recently published collections. The publication in 1965 of the correspondence between Euler and Christian Goldbach has been mentioned earlier.

It is significant that the first volume, A1, published in the fourth series of Euler's *Opera omnia*, contains a complete list of all existing letters to and from Euler (about 3,000), together with a summary of their contents. Volume A5 of this series (1980), edited by A. P. Juškevič and R. Taton, contains Euler's correspondence with A. C. Clairaut, J. d'Alembert, and J. L. Lagrange.

The correspondence between Euler and Lagrange from 1754 to 1775 gives valuable testimony to the development of personal relations between two of the most important scientists of that time. The letter exchange begins with a letter from the 18-year-old Lagrange, who lived in Turin, containing a query in which he mentions the analogy in the development of the binomial $(a+b)^m$ and the differential $d^m(xy)$. Mathematically isolated, Lagrange expresses his admiration for Euler's work, particularly in mechanics. Especially significant is the second letter to Euler (1755). In it Lagrange announces, without details, his new methods in the calculus of variations; Euler at once notes the advantage of these methods over the ones in his *Methodus inveniendi lines curvas maximi minimive proprietate gaudentes* (1744), and heartily congratulates Lagrange. In 1756 Lagrange develops the differential calculus for several variables and investigates, for the first time, minimal surfaces. After an interruption of three years, Lagrange continues the correspondence by sending his work *La nature et la propagation du son*, and we find interesting discussions on the problem of vibrating strings, which had been carried on since 1749 between d'Alembert, Euler and Daniel Bernoulli.

After a lengthy pause, Euler resumes the correspondence. The first letter (1765) concerns the discussion with d'Alembert on vibrating strings, and the librations of the moon. In a second, Euler tells Lagrange that he has been granted permission by Friedrich II to return to Petersburg, and is attempting to have Lagrange come there. In later correspondence, the emphasis is on questions in the theory of numbers and in algebra. Pell's equation $x^2 - ay^2 = b$, and in particular $p^2 - 13q^2 = 101$, are discussed. Other topics deal with arithmetic, questions concerning developable surfaces, and the motion of the moon.

In 1770 Lagrange writes of his plan to publish Euler's *Algebra* in French, and to add to it an appendix; the published book is mailed on July 13, 1773. The last of Euler's letters, dated March 23, 1775, is remarkable by the exceptionally warm congratulations for Lagrange's work, especially about elliptic integrals. It may be conjectured that this was not the end of the correspondence, but unfortunately no additional letters have survived.

Postscript

This overview of Euler's life and work touches only a small part of the wealth of material to be found in the scholarly essays in the Basel memorial volume. In addition to careful and detailed analysis of many of Euler's scientific and mathematical achievements, these chapters contain new information on all aspects of Euler's private and academic life, his family, his philosophical and religious views, and the fabric of his life and work at the St. Petersburg Academy. In view of the overwhelming volume and diversity of Euler's work, it may never be possible to produce a comprehensive scientific biography of his genius. It is to be hoped that these newest contributions to the study of his life and work will provide impetus for further study and publication of many of the yet unpublished papers which are the unknown legacy of this mathematical giant.

The author and the editor express deep appreciation to Branko Grünbaum, who translated from the German the author's original manuscript. Doris Schattschneider took the trouble to shorten this manuscript from 38 to 22 pages.

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Opposite side of medallion shown on p. 260 depicts the Academy of Sciences building in Leningrad (formerly St. Petersburg) where Euler worked for a great part of his life.

Ars Expositionis: Euler as Writer and Teacher

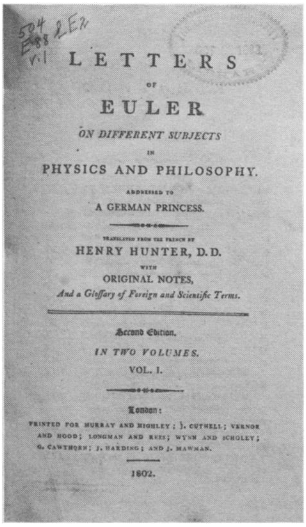
G. L. ALEXANDERSON
University of Santa Clara
Santa Clara, CA 95053

In a recent article by Harold M. Edwards, we are exhorted to "read the masters!" [**2**]. While we can all agree that in general this is good advice, we may still be reluctant to go to Newton's *Principia* in search of a lucid treatment of the calculus or to Gauss' *Disquisitiones Arithmeticae* for an account of some famous theorems of number theory and how they came to be proved. One need have no such reluctance, however, about reading Euler. Among the masters he stands out for the clarity of his writing, his willingness to show how he came to his discoveries, and his open admission of failure to prove a conjecture for which he had convincing evidence. George Pólya sums it up when he writes (italics ours):

> A master of inductive research in mathematics, he [Euler] made important discoveries (on infinite series, in the Theory of Numbers, and in other branches of mathematics) by *induction*, that is, by *observation*, *daring guess*, and *shrewd verification*. In this respect, however, Euler is not unique; other mathematicians, great and small, used induction extensively in their work.
>
> Yet Euler seems to me almost unique in one respect: he takes pains to present the relevant inductive evidence carefully, in detail, in good order. He presents it convincingly but honestly, as a genuine scientist should do. His presentation is "the candid exposition of the ideas that led him to those discoveries" and has a distinctive charm. Naturally enough, as any other author, he tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself [**10**, p. 90].

Even among his contemporaries, Euler was known as someone whose writings were particularly clear and elegant. Nicholas Fuss, his assistant in St. Petersburg and a fellow citizen of Basel, wrote in his eulogy on the death of Euler in 1783 [**7**, p. 14] about the clarity of his ideas, the precision in their statement and the order in which they were arranged. The son of Nicholas Fuss, Paul-Heinrich Fuss, in his preface to the collection of letters he edited in 1843 [**7**, p. xli] wrote of Euler's "remarkably simple and lucid exposition of his profound research and his skillful choice of examples."

Vollständige
Anleitung
zur
Algebra
von
Hrn. Leonhard Euler.
Erster Theil.
Von den verschiedenen Rechnungs-Arten, Verhältnißen und Proportionen.
St. Petersburg.
gedruckt bey der Kayß. Acad. der Wissenschaften 1770.



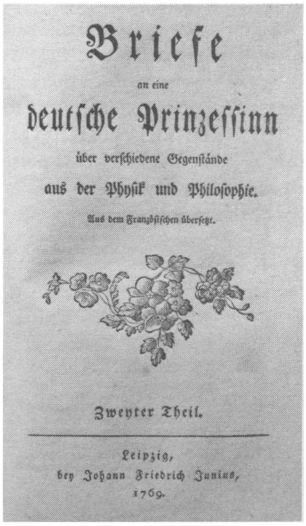
LETTERS
OF
EULER
ON DIFFERENT SUBJECTS
IN
PHYSICS AND PHILOSOPHY.
ADDRESSED TO
A GERMAN PRINCESS.

TRANSLATED FROM THE FRENCH BY
HENRY HUNTER, D.D.
WITH
ORIGINAL NOTES,
And a Glossary of Foreign and Scientific Terms.

Second Edition.
IN TWO VOLUMES.
VOL. I.

London:
PRINTED FOR MURRAY AND HIGHLEY; J. CUTHELL; VERNOR AND HOOD; LONGMAN AND REES; WYNN AND SCHOLEY; G. CAWTHORN; J. HARDING; AND J. MAWMAN.

1802.



Briefe
an eine
deutsche Prinzessinn
über verschiedene Gegenstände
aus der Physik und Philosophie.
Aus dem Französischen übersetzt.

Zweyter Theil.

Leipzig,
bey Johann Friedrich Junius,
1769.

Euler's career did not involve the teaching that many mathematicians have done routinely over the years, since he spent his professional life at the Imperial Academy of Sciences in St. Petersburg and the Royal Society of Sciences in Berlin. He nevertheless seems to have had remarkable success at teaching on those occasions when he took on students. N. Fuss tells the story of the young tailor's apprentice Euler brought back to St. Petersburg with him from Berlin in the role of a domestic servant and "who had no smattering of mathematics" but who was the writer to whom Euler dictated his textbook *Vollständige Anleitung der Algebra*, "as generally admired for the circumstances in which it was composed as for the supreme degree of clarity and of method that prevails throughout. The creative spirit reveals itself even in this purely elementary work" [7, p. 50]. Du Pasquier tells that Euler's son, Johann Albrecht Euler, claimed that by having the text of the algebra dictated to the young servant the boy "became capable of solving by himself even difficult algebraic problems, without need of any help!" [9, p. 113]. The translator of the algebra text into English wrote that

> Here, all is luminous, easy, and obvious. In giving the most difficult demonstrations, and in illustrating the most abstruse subjects, the different steps of the rationale are so many axioms; and it was Euler's great talent to render their order and dependence, in their progress through the mind, clear and evident to the meanest capacity [3, pp. v–vi].

How could one help but learn?

Of course, Euler wrote other texts, the well-known text on the differential calculus (*Institutiones calculi differentialis*) and later his three volume set on the integral calculus (*Institutiones calculi integralis*). These classics set the topics for calculus texts for many years and, again, are known for their clarity and the "choice of examples, as numerous as they are instructive" [9, p. 114]. Perhaps his most famous teaching book is, however, the *Lettres à une Princesse d'Allemagne*. This book, written for the instruction of the 15 year-old future Princess of Anhalt-Dessau, took up topics in mechanics, astronomy, physics, optics, and acoustics "with a marvelous clarity" [9, p. 59]. This set of volumes which appeared in 1768 in St. Petersburg was an immediate success throughout Europe, though it may have been written at a level quite beyond that of a 15 year-old. It was translated into Russian, and appeared in four editions. In Paris, in Leipzig, and in Bern, du Pasquier tells us, there were French editions, twelve in all. They were issued in English nine times, in German six times, in Dutch twice, in Swedish twice, and there were also translations into Italian, Danish and Spanish. The first translator of the *Lettres* into English, Henry Hunter, wrote in his preface

It was long a matter of surprise to me, that a work so well known, and so justly esteemed, over the whole European Continent, as Euler's *Letters to a German Princess*, should never have made its way into our Island, in the language of the Country. While Petersburg, Berlin, Paris, nay the capital of every petty German principality, was profiting by the ingenious labors of this amiable man, and acute philosopher, the name of Euler was a sound unknown to the ear of youth in the British metropolis. I was mortified to reflect that the specious and seductive productions of a *Rousseau*, and the poisonous effusions of a *Voltaire*, should be in the hands of so many young men, not to say young women, to the perversion of the understanding, and the corruption of the moral principle, while the simple and useful instructions of the virtuous Euler were hardly mentioned.

As soon as Providence had bestowed on me the blessing of children, I felt it to be my duty to charge myself with their instruction. How I have succeeded it becomes not me to say: but every day I live, the importance of early and proper culture is more deeply impressed on my mind....

The subjects of these letters, and the author's method of treating them, seem to me much adapted to this purpose. With the assistance of a very moderate apparatus, they might conduct youth of both sexes, with equal delight and emolument, to a very competent knowledge of natural philosophy: very little previous elementary knowledge is necessary to a profitable perusal of them, and that little may be very easily acquired.

A considerable part of our common school education, it is well known, consists of the study of the elegant and amusing poetical fictions of antiquity. Without meaning to decry this, may I not be permitted to hint, that it might be of importance frequently to recall young minds from an ideal world, and its ideal inhabitants, to the real world, of which they are a part, and of which it is a shame to be ignorant. [**4**; xiii-xvi]

As in the other writings of Euler that were intended to instruct, there was much greater content than one often sees in textbooks, and the content of the *Lettres* is still being discussed today [**1**].

At least one other book of Euler's might be considered a book for instruction, the *Introductio in analysin infinitorum* (1748). Though this is not really a text, it is, nevertheless, a compilation of known work in analysis together with a good bit of new material supplied by Euler. It is one of Euler's most delightful and rewarding works, "as marvellous in its clarity of exposition as for the richness of its contents" [**9**, p. 113]. The first volume contains a lengthy discussion on the correct definition of a function, but much of the two-volume set is devoted to the solution of wonderful problems. For example, you can find here his argument that the series of the reciprocals of the squares of consecutive integers sums to $\pi^2/6$ and the evaluation of the zeta function for other even integral arguments, the introduction of the theory of partitions, and many properties of logarithms, exponentials and other functions that we now have come to take for granted in courses in classical analysis.

What sets Euler apart from other great masters who wrote mathematics, including textbooks, many of whom even wrote very clearly? As Pólya has noted, part of the answer probably lies in Euler's approach to mathematics and the kind of mathematics that he did so well. He used a great deal of induction in his work, for pattern recognition and discovery of formulas were his forte. Unlike others—Gauss' name comes to mind—Euler did not attempt to hide the origins of his theorems, but, on the contrary, he went out of his way to motivate them by giving many examples. Pólya quotes Condorcet as saying that

> He [Euler] preferred instructing his pupils to the little satisfaction of amazing them. He would have thought not to have done enough for science if he should have failed to add to the discoveries, with which he enriched science, the candid exposition of the ideas that led him to those discoveries [**10**, p. 90].

Euler admits to having trouble proving certain conjectures, but then proceeds to use unproved results, cautioning the reader that certain steps are not yet completely proved. In reading Euler we can see a great, creative mind at work. Further, it is encouraging to us lesser creatures to read of Euler's struggles to discern patterns and to prove his conjectures. His account of his discovery of what is now termed the pentagonal number theorem (which for years defied his attempts at proof) was chosen by Pólya as exemplary of Euler's style of exposition. George Andrews gives a full modern exposition of Euler's proof of this theorem (this *Magazine*, pp. 279–284); here we are interested in Euler's own early exposition of the result. A full account of Euler's presentation, taken from [4] and translated by Pólya, can be found in [**10**, pp. 91–98]. Here we extract a few sections to demonstrate Euler's style.

In discussing the function $\sigma(n)$, which gives the sum of the divisors of n, he begins

> Till now the mathematicians tried in vain to discover some order in the sequence of the prime numbers and we have every reason to believe that there is some mystery which the human mind shall never penetrate. To convince oneself, one has only to glance at the tables of the primes, which some people took the trouble to compute beyond a hundred thousand, and one perceives that there is no order and no rule. This is so much more surprising as the arithmetic gives us definite rules with the help of which we can continue the sequence of the primes as far as we please, without noticing, however, the least trace of order. I am myself certainly far from this goal, but I just happened to discover an extremely strange law governing the sums of the divisors of the integers which, at the first glance, appear just as irregular as the sequence of the primes, and which, in a certain sense, comprise even the latter. This law, which I shall explain in a moment, is, in my opinion, so much more remarkable as it is of such a nature that we can be assured of its truth without giving it a perfect demonstration. Nevertheless, I shall present such evidence for it as might be regarded as almost equivalent to a rigorous demonstration.

He proceeds to develop the formula,

$$\sigma(n)=\sum_{j=1}^{\infty}(-1)^{j+1}\sigma(n-n_j),$$

where

$$n_j=\tfrac{1}{2}j(3j\pm 1)\text{ and }\sigma(0)=n,$$

through the use of many examples and gives a heuristic argument which, unfortunately, depends on the infinite product formula

$$\prod_{k=1}^{\infty}(1-x^k)=\sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n+1)/2}$$

which he is unable to prove. But at each step he tells what led him to the next discovery and wonders at the beautiful patterns. He says at one point,

> The examples that I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula. Now, this beautiful property of the numbers is so much more surprising as we do not perceive any intelligible connection between the structure of my formula and the nature of the divisors with the sum of which we are here concerned.

Later, he writes,

> I confess that I did not hit on this discovery by mere chance, but another proposition opened the path to this beautiful property—another proposition of the same nature which must be accepted as true although I am unable to prove it. And although we consider here the nature of integers to which the Infinitesimal Calculus does not seem to apply, nevertheless I reached my conclusion by differentiations and other devices. I wish that somebody would find a shorter and more natural way, in which the consideration of the path that I followed might be of some help, perhaps.

In reading Euler's exposition, one cannot help but agree with Pólya that from it we can learn "a great deal about mathematics, or the psychology of invention, or inductive reasoning" [**10**, p. 99]. His techniques as well as his results are a bountiful source of ideas for modern researchers. Several authors in this issue of the *Magazine* indicate by their discussion of some aspect of Euler's work how his exposition is extraordinarily rich in ideas. As a final example, we note that Euler uses inductive reasoning similar to that described above and a daring, though incorrect, use of analogy to evaluate $\zeta(2)$. In doing so, he seems to be talking to the reader, explaining, sometimes apologizing for the lack of rigor, but always giving insights into the process of discovery [6], [10, pp. 17–21]. A recent paper [11] focuses on Euler's method used on $\zeta(2)$ and shows its fruitfulness.

George Pólya has long been an advocate of Euler's style of presenting mathematics. The author wishes to thank Professor Pólya for his helpful suggestions in the preparation of this note.

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Marble bust of Leonhard Euler, sculpted in 1875 by Heinrich Ruf. The original is in the lobby of the "Bernoullianum," today the Geographical Institute of the University of Basel.

Euler's Pentagonal Number Theorem

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One of Euler's most profound discoveries, the *Pentagonal Number Theorem* [7], has been beautifully described by André Weil:

> Playing with series and products, he discovered a number of facts which to him looked quite isolated and very surprising. He looked at this infinite product
>
> $$(1-x)(1-x^2)(1-x^3)\cdots$$
>
> and just formally started expanding it. He had many products and series of that kind; in some cases he got something which showed a definite law, and in other cases things seemed to be rather random. But with this one, he was very successful. He calculated at least fifteen or twenty terms; the formula begins like this:
>
> $$\Pi(1-x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15}\cdots$$
>
> where the law, to your untrained eyes, may not be immediately apparent at first sight. In modern notation, it is as follows:
>
> $$\prod_{1}^{\infty}(1-q^n) = \sum_{-\infty}^{+\infty}(-1)^n q^{n(3n+1)/2} \qquad (1)$$
>
> where I've changed x into q since q has become the standard notation in elliptic function-theory since Jacobi. The exponents make up a progression of a simple nature. This became immediately apparent to Euler after writing down some 20 terms; quite possibly he calculated about a hundred. He very reasonably says, "this is quite certain, although I cannot prove it;" ten years later he does prove it. He could not possibly guess that both series and product are part of the theory of elliptic modular functions. It is another tie-up between number-theory and elliptic functions [22, pp. 97–98].

G. Pólya [16, pp. 91–98] provides a more extensive account of Euler's wonderful discovery together with a translation of Euler's own description [6].

The numbers $n(3n-1)/2$ are called "pentagonal numbers" because of their relationship to pentagonal arrays of points. Figure 1 illustrates this. Legendre [14, pp. 131–133] observed that purely formal multiplication of the terms on the left side of (1) produces the term $\pm q^N$ precisely as often as N is representable as a sum of distinct positive integers; the "+" is taken when there is an even number of summands in the representation and the "−" when the number of summands is odd. For example,

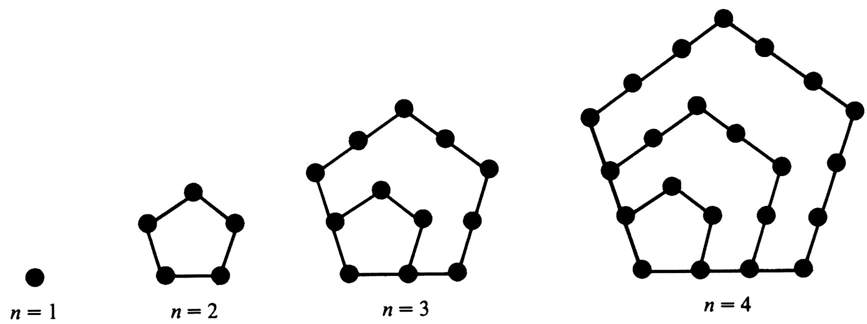

Figure 1. **The first four pentagonal numbers are 1, 5, 12, 22.**

$$
\begin{aligned}
&(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots \\
&\quad = 1 - q - q^2 - q^3 - q^4 + q^{1+2} + q^{1+3} + q^{1+4} + q^{2+3} \\
&\qquad + q^{2+4} + q^{3+4} - q^{1+2+3} - q^{1+2+4} - q^{1+3+4} - q^{2+3+4} + q^{1+2+3+4} + \cdots .
\end{aligned}
$$

The term **partition** is usually used to describe the representation of a positive integer as the sum of positive integers. In this article, we are concerned with *unordered* partitions; two such partitions are considered the same if the terms in the sum are the same, e.g., $1+2$ and $2+1$ are considered as the same partition of 3. Thus Legendre's observation may be matched up with the actual infinite series expansion (1) as follows.

THEOREM. *Let $p_e(n)$ denote the number of partitions of n into an even number of distinct summands. Let $p_o(n)$ denote the number of partitions of n into an odd number of distinct summands. Then*

$$
p_e(n) - p_o(n) = \begin{cases} (-1)^j & \text{if } n = j(3j \pm 1)/2 \\ 0 & \text{otherwise.} \end{cases} \tag{2}
$$

The impact of Euler's Pentagonal Number Theorem and Legendre's observations on subsequent developments in number theory is enormous. Both (1) and (2) are justly famous. Indeed, F. Franklin's purely arithmetic proof of (2) [**10**] (see also [**21**, pp. 261–263]) has been described by H. Rademacher as the first significant achievement of American mathematics. Franklin's proof is so elementary and lovely that it has been presented many times over in elementary algebra and number theory texts [**5**, pp. 563–564], [**11**], [**12**, pp. 206–207], [**13**, pp. 286–287], [**15**, pp. 221–222].

It is, however, interesting to note that Euler's proof of (1) alluded to by Weil remains almost unknown. In recent years only Rademacher's book has contained an exposition of it [**17**, pp. 224–226]. This book and earlier books [**4**, pp. 23–24] have presented it more or less as Euler did. While the idea behind Euler's proof is ingenious (as one would expect), the mathematical notation of Euler's day hides the fact that other results of significance are either transparent corollaries of Euler's proof or lie just below the surface. The remainder of this article is devoted to a long overdue modern exposition of Euler's proof and an examination of its consequences.

To begin, we define a function of two variables:

$$
f(x,q) = 1 - \sum_{n=1}^{\infty} (1-xq)(1-xq^2)\cdots(1-xq^{n-1})x^{n+1}q^n. \tag{3}
$$

(Absolute convergence of all series and products considered is ensured by $|q| < 1$, $|x| < |q|^{-1}$). We first note that

$$
f(1,q) = \prod_{n=1}^{\infty} (1-q^n). \tag{4}
$$

This is because we may easily establish the identity

$$
1 - \sum_{n=1}^{N} (1-q)(1-q^2)\cdots(1-q^{n-1})q^n = \prod_{n=1}^{N} (1-q^n) \tag{5}
$$

by mathematical induction on N (a nice exercise for the reader). Thus (4) is the limiting case of (5) as $N \to \infty$.

The main step in Euler's proof is essentially the verification of the following functional equation:

$$
f(x,q) = 1 - x^2q - x^3q^2 f(xq,q). \tag{6}
$$

Actually Euler does equation (6) over and over, first with $x = 1$, then $x = q$, then $x = q^2$, and so on [7, pp. 473–475]; page 473 of his paper in *Opera Omnia* is shown on the next page. This repetition of special cases of (6) tends to hide what exactly is happening. To prove (6), we take the defining equation (3) through a sequence of algebraic manipulations:

$$\begin{aligned}
f(x,q) &= 1 - x^2q - \sum_{n=2}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{n-1})x^{n+1}q^n \\
&= 1 - x^2q - \sum_{n=1}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1} \\
&= 1 - x^2q - \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1}(1-xq) \\
&= 1 - x^2q - \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1} \\
&\quad + \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2} \\
&= 1 - x^2q - x^3q^2 - \sum_{n=2}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1} \\
&\quad + \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2} \\
&= 1 - x^2q - x^3q^2 - \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n+1})x^{n+3}q^{n+2} \\
&\quad + \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2} \\
&= 1 - x^2q - x^3q^2 - \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2}\{(1-xq^{n+1})-1\} \\
&= 1 - x^2q - x^3q^2\left\{1 - \sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+1}q^{2n+1}\right\} \\
&= 1 - x^2q - x^3q^2f(xq,q).
\end{aligned}$$

If you followed the above sequence of steps carefully, you see how at each stage things seem to fit together magically at just the right moment. Also you can appreciate the complication of repeated presentation of the same steps first with $x = 1$, then $x = q$, then $x = q^2$, etc. However, the empirical discovery of (6) by Euler must have come precisely in this repetitive manner.

The rest of Euler's proof is now almost mechanical. Equation (6) is repeatedly iterated; thus

$$\begin{aligned}
f(x,q) &= 1 - x^2q - x^3q^2\left(1 - x^2q^3 - x^3q^5f(xq^2,q)\right) \\
&= 1 - x^2q - x^3q^2 + x^5q^5 + x^6q^7\left(1 - x^2q^5 - x^3q^8f(xq^3,q)\right) \\
&\ \ \vdots \\
&= 1 + \sum_{n=1}^{N-1}(-1)^n\left(x^{3n-1}q^{n(3n-1)/2} + x^{3n}q^{n(3n+1)/2}\right) \\
&\quad + (-1)^N x^{3N-1}q^{N(3N-1)/2} + (-1)^N x^{3N}q^{N(3N+1)/2}f(xq^N,q),
\end{aligned} \tag{7}$$

47—48] EVOLUTIO PRODUCTI INFINITI $(1-x)(1-xx)(1-x^3)(1-x^4)$ etc. 473

2. Quoniam hi termini omnes factorem habent communem $1-x$, eo evoluto singuli termini discerpentur in binas partes, quas ita repraesentemus

$$\begin{aligned} A = {} & xx + x^3(1-xx) + x^4(1-xx)(1-x^3) + x^5(1-xx)(1-x^3)(1-x^4) + \text{etc.} \\ & - x^3 - x^4(1-xx) - x^5(1-xx)(1-x^3) - x^6(1-xx)(1-x^3)(1-x^4) - \text{etc.} \end{aligned}$$

Hinc iam binae partes eadem potestate ipsius x affectae in unam contrahantur ac resultabit pro A sequens forma

$$A = xx - x^5 - x^7(1-xx) - x^9(1-xx)(1-x^3) - x^{11}(1-x^2)(1-x^3)(1-x^4) - \text{etc.},$$

ubi duo termini primi $xx - x^5$ iam sunt evoluti; sequentes autem procedunt per has potestates $x^7, x^9, x^{11}, x^{13}, x^{15}$, quarum exponentes binario crescunt.

3. Ponamus nunc simili modo ut ante

$$A = xx - x^5 - B,$$

ita ut sit

$$B = x^7(1-xx) + x^9(1-xx)(1-x^3) + x^{11}(1-xx)(1-x^3)(1-x^4) + \text{etc.},$$

cuius omnes termini habent factorem communem $1-xx$, quo evoluto singuli termini in binas partes discerpantur, uti sequitur,

$$\begin{aligned} B = {} & x^7 + x^9(1-x^3) + x^{11}(1-x^3)(1-x^4) + x^{13}(1-x^3)(1-x^4)(1-x^5) + \text{etc.} \\ & - x^9 - x^{11}(1-x^3) - x^{13}(1-x^3)(1-x^4) - x^{15}(1-x^3)(1-x^4)(1-x^5) - \text{etc.} \end{aligned}$$

Hic iterum bini termini, qui eandem potestatem ipsius x habent praefixam, in unam colligantur et prodibit

$$B = x^7 - x^{12} - x^{15}(1-x^3) - x^{18}(1-x^3)(1-x^4) - x^{21}(1-x^3)(1-x^4)(1-x^5) - \text{etc.}$$

ubi iam potestates ipsius x crescunt ternario.

4. Statuatur nunc porro

$$B = x^7 - x^{12} - C,$$

ita ut sit

$$C = x^{15}(1-x^3) + x^{18}(1-x^3)(1-x^4) + x^{21}(1-x^3)(1-x^4)(1-x^5) + \text{etc.},$$

LEONHARDI EULERI Opera omnia I3 Commentationes arithmeticae 60

another fine exercise in mathematical induction on N. In the limit as $N \to \infty$ we find

$$f(x,q) = 1 + \sum_{n=1}^{\infty} (-1)^n \left(x^{3n-1} q^{n(3n-1)/2} + x^{3n} q^{n(3n+1)/2} \right). \tag{8}$$

Therefore by (4) and (8),

$$\begin{aligned} \prod_{n=1}^{\infty} (1-q^n) = f(1,q) &= 1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}, \end{aligned}$$

which completes the proof of (1).

It should be noted that several authors (L. J. Rogers [**18**, pp. 334–335], G. W. Starcher [**19**], and N. J. Fine [**9**]) also found formula (8) essentially in the way we have; however, none has noted that he was, in fact, rediscovering Euler's proof in simpler clothing. M. V. Subbarao [**20**] has also shown the connection between (8) and Franklin's arithmetic proof [**10**].

Now the reader may naturally ask: Was anything gained in this general formulation of Euler's proof besides simplicity of presentation? Is any new information available that was hidden before? We can answer a resounding YES just by setting $x=-1$ in (3) and (8). This substitution gives the equation

$$\begin{aligned}1+\sum_{n=1}^{\infty}(1+q)(1+q^2)\cdots(1+q^{n-1})(-1)^n q^n \\ =f(-1,q)=1+\sum_{n=1}^{\infty}\left(-q^{n(3n-1)/2}+q^{n(3n+1)/2}\right).\end{aligned}\tag{9}$$

Equation (9) yields a corollary as appealing as Legendre's Theorem. It implies that *the product*

$$(1+q)(1+q^2)\cdots(1+q^{n-1})q^n$$

when multiplied out produces the term q^N exactly as often as N can be partitioned into distinct summands with largest part equal to n. For example, when $n=4$,

$$\begin{aligned}(1+q)(1+q^2)(1+q^3)q^4=q^4+q^{4+1}+q^{4+2}+q^{4+3}+q^{4+2+1}\\ +q^{4+3+1}+q^{4+3+2}+q^{4+3+2+1}\end{aligned}$$

Hence the series on the left side of (9) when expanded out yields the term $\pm q^N$ for each partition of N into distinct summands; the "+" occurs if the largest summand is even, and the "−" occurs if the largest summand is odd. In the same manner that (1) yielded Legendre's Theorem, we see that (9) yields the following equally elegant but little publicized result found by N. J. Fine [**8**] more than 118 years after Legendre's observation.

THEOREM. *Let $\pi_e(n)$ denote the number of partitions of n with distinct summands the largest of which is even. Let $\pi_o(n)$ denote the number of partitions of n with distinct summands the largest of which is odd. Then*

$$\pi_e(n)-\pi_o(n)=\begin{cases}1 & \text{if } n=j(3j+1)/2,\\ -1 & \text{if } n=j(3j-1)/2,\\ 0 & \text{otherwise}.\end{cases}$$

Let us check out the theorems with an example. The partitions of $n=12$ into distinct parts are: 12, 11 + 1, 10 + 2, 9 + 3, 9 + 2 + 1, 8 + 4, 8 + 3 + 1, 7 + 5, 7 + 4 + 1, 7 + 3 + 2, 6 + 5 + 1, 6 + 4 + 2, 6 + 3 + 2 + 1, 5 + 4 + 3, 5 + 4 + 2 + 1. The partitions enumerated by each of $p_e(12)$, $p_o(12)$, $\pi_e(12)$ and $\pi_o(12)$ are listed in the following table:

$p_e(12)$	$p_o(12)$	$\pi_e(12)$	$\pi_o(12)$
11 + 1	12	12	11 + 1
10 + 2	9 + 2 + 1	10 + 2	9 + 3
9 + 3	8 + 3 + 1	8 + 4	9 + 2 + 1
8 + 4	7 + 4 + 1	8 + 3 + 1	7 + 5
7 + 5	7 + 3 + 2	6 + 5 + 1	7 + 4 + 1
6 + 3 + 2 + 1	6 + 5 + 1	6 + 4 + 2	7 + 3 + 2
5 + 4 + 2 + 1	6 + 4 + 2	6 + 3 + 2 + 1	5 + 4 + 3
	5 + 4 + 3		5 + 4 + 2 + 1.

Thus $p_e(12)-p_o(12)=7-8=-1$ and $\pi_e(12)-\pi_o(12)=7-8=-1$, as predicted by our theorems.

Beyond this immediate pleasant discovery that Euler's approach, properly modernized, yields Fine's Theorem, we may ask: Are there interesting extensions of Euler's method that yield more than equation (8)? Here again the answer is positive. L. J. Rogers [**18**, p. 334], apparently unaware of how Euler's proof worked, showed in almost precisely the way we proved (8) that

$$1+\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-b)(1-bq)\cdots(1-bq^{n-1})}=\frac{1-at}{1-t}+$$

$$\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})\left(1-\frac{atq}{b}\right)\left(1-\frac{atq^2}{b}\right)\cdots\left(1-\frac{atq^n}{b}\right)b^n t^n q^{n^2-n}(1-atq^{2n})}{(1-b)(1-bq)\cdots(1-bq^n)(1-t)(1-tq)\cdots(1-tq^{n+1})}. \tag{10}$$

Rogers in fact showed that if $f(a,b,t)$ denotes the left side of (10), then

$$f(a,b,t)=\frac{1-at}{1-t}+t\frac{(1-a)(b-atq)}{(1-b)(1-t)}f(aq,bq,tq). \tag{11}$$

This result and deeper extensions of it that require much more than Euler's method have had a major impact in the theory of partitions [**1**, Secs. 3 and 4], [**2**, Ch. 7], [**3**, Ch. 3]. N. J. Fine [**9**] also independently rediscovered (10).

Surely the story unfolded here emphasizes how valuable it is to study and understand the central ideas behind major pieces of mathematics produced by giants like Euler. The discoveries of theorems as appealing as the two we have described would not be separated by 118 years if students of additive number theory had followed this advice.

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Euler and Quadratic Reciprocity

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In a letter to Goldbach bearing the date 28 August 1742, Euler described a property of positive whole numbers that was to play a central role in the history of the theory of numbers. (The original is a mixture of Latin and German, which I have translated into English as best I can. The letter can be found in [**1**] or [**3**].)

> Whether there are series of numbers which either have no divisors of the form $4n+1$, or which even are prime, I very much doubt. If such series could be found, however, one could use them to great advantage in finding prime numbers.
>
> By the way, the prime divisors of all series of numbers which are given by the formula $\alpha xx \pm \beta yy$ show a very orderly pattern which, although I have no demonstration of it as yet, seems to be completely correct. For this reason I take the liberty of communicating to Your Excellency a few such theorems; from these, infinitely many others can be derived.
>
> I. If x and y are relatively prime, the formula $xx+yy$ has no prime divisors other than those contained in the form $4n+1$, and these prime numbers are themselves all contained in the form $xx+yy$. I put this known theorem at the beginning in order to make the connection of the others more apparent.
>
> II. The formula $2xx+yy$ has no prime divisors other than those contained in the form $8n+1$ or $8n+3$. And whenever $8n+1$ or $8n+3$ is prime, it is the sum of a square and twice a square, that is, it is of the form $2xx+yy$.
>
> III. The formula $3xx+yy$ has no prime divisors other than those contained in the forms $12n+1$ and $12n+7$ (or the single form $6n+1$). And whenever $6n+1$ is a prime number it is contained in the form $3xx+yy$.
>
> IV. The formula $5xx+yy$ has no prime divisors other than those contained in the forms $20n+1$, $20n+3$, $20n+9$, $20n+7$, and every prime number contained in one of these four forms is itself a number of the form $5xx+yy$.
>
> V. The formula $6xx+yy$ has no prime divisors other than those contained in one of the four forms $24n+1$, $24n+5$, $24n+7$, $24n+11$, and every prime number contained in one of these forms is itself a number of the form $6xx+yy$.
>
> VI. The formula $7xx+yy$ has no prime divisors other than those contained in one of the 6 forms $28n+1$, $28n+9$, $28n+11$, $28n+15$, $28n+23$, $28n+25$ (or in one of the three $14n+1$, $14n+9$, $14n+11$), and every prime number contained in one of these forms is itself a number of the form $7xx+yy$.
>
> From this it is thus clear that the expression $pxx+yy$ can have no prime divisors other than those contained in a certain number of forms of the type $4pn+s$, where s represents some numbers which, although they appear to have no particular order, actually proceed according to a very beautiful rule, which is clarified by these theorems:
>
> VII. If a prime number of the form $4pn+s$ is a divisor of the formula $pxx+yy$ then likewise every prime number contained in the general form $4pn+s^k$ will be a divisor of the formula $pxx+yy$ and indeed will itself be a number of the form $pxx+yy$. For example, because a prime number $28n+9$ is a number of the form $7xx+yy$ [$37=\text{prime}=28\cdot 1+9=7\cdot 4+9$] prime numbers $28n+81$ $(28n+25)$ and $28n+729$ $(28n+1)$ are indeed numbers of the form $7xx+yy$ [53 and 29].
>
> VIII. If two prime numbers $4pn+s$ and $4pn+t$ are divisors of the formula $pxx+yy$ then every prime number of the form $4pn+s^kt^i$ is also a number of the form $pxx+yy$.
>
> Thus when one has found a few prime divisors of such an expression $pxx+yy$ one can easily find all possible divisors using these theorems. For example, let $13xx+yy$ be the given formula, which includes the numbers 14, 17, 22, 29, 38, 49, 62, etc. Thus 1, 7, 11, 17, 19, 29, 31

are prime numbers which divide the formula $13xx+yy$. Therefore all prime numbers of the forms $52n+1$, $52n+7$, $52n+11$ etc. can be divisors of $13xx+yy$. But the formula $52n+7$ gives, by Theorem VII, also these $52n+49$, $52n+343$ (or $52n+31$), $52n+7\cdot 31$, or $52n+9$, further $52n+7\cdot 9$, or $52n+11$, further $52n+7\cdot 11$, or $52n+25$, further $52n+7\cdot 25$, or $52n+19$, further $52n+7\cdot 19$, or $52n+29$, further $52n+7\cdot 29$, or $52n+47$, further $52n+7\cdot 47$, or $52n+17$, further $52n+7\cdot 17$, or $52n+15$, further $52n+7\cdot 15$, or $52n+1$ and at this point the numbers cease to be different which when added to $52n$ give prime numbers of the form $13xx+yy$. Thus from the single fact that 7 can be a divisor of the form $13xx+yy$ the last two theorems imply that all prime numbers of any of the forms

$52n+1$;	$52n+31$;	$52n+25$;	$52n+47$
$52n+7$;	$52n+9$;	$52n+19$;	$52n+17$
$52n+49$;	$52n+11$;	$52n+29$;	$52n+15$

have the form $13xx+yy$ and also can be divisors of such numbers $13xx+yy$, and also more formulas can not be derived using the theorems. From this it is known that no prime number can be a divisor of the form $13xx+yy$ other than those contained in the 12 formulas that have been found. Now every prime number of the form $4pn+1$ can be a divisor of $pxx+yy$. From this, beautiful properties can be derived, as, for example, because 17 is prime and also of the form $2xx+yy$ it follows that whenever $17^m \pm 8n$ is prime it must also be of the form $2xx+yy$. And when $17^m \pm 8n$ is a number of the form $2xx+yy$ which admits no divisors of this form, it is certainly a prime number.

The same situation occurs with the divisors of the forms $pxx-yy$ or $xx-pyy$, which, when they are prime, must be contained in the form $4np \pm s$, where s represents certain determined numbers. Namely, in a few cases, one will have

1. All prime divisors of the form $xx-yy$ contained in the form $4n \pm 1$, which is clear.
2. All prime divisors of the form $2xx-yy$ contained in the form $8n \pm 1$.

Coroll. Therefore a prime number of the form $8n \pm 3$ is not a number of the form $2xx-yy$.

3. All prime divisors of the form $3xx-yy$ contained in the form $12n \pm 1$.
4. All prime divisors of the form $5xx-yy$ contained in either the form $20n \pm 1$ or the form $20n \pm 9$ (or in the single one $10n \pm 1$).

etc.

And if a prime number $4pn+s$ divides the form $pxx-yy$ or $xx-pyy$, then $\pm 4np \pm s^k$ will itself be of the form $pxx-yy$ or $xx-pyy$, whenever it is prime. If two prime numbers s and t are numbers of the form $pxx-yy$, then whenever $4np \pm s^\mu t^\nu$ is prime it will also be a number of the form $pxx-yy$. Thus, because 7 and 17 are prime numbers and of the form $2xx-yy$, $\pm 8n \pm 7^\mu \cdot 17^\nu$ will also be of this form whenever it is prime. Let $\mu=1$, $\nu=1$, so $7\cdot 17=119$ and $119+8=127=$ prime, and consequently $127=2xx-yy=2\cdot 64-1$. From this it is now clear that it is not possible to find sequences of numbers of the type $pxx \pm qyy$ which do not admit divisors of the form $4n+1$.

But I am convinced that I have not exhausted this material, rather, that there are countless wonderful properties of numbers to be discovered here, by means of which the theory of divisors could be brought to much greater perfection; and I am convinced that if Your Excellency were to consider this subject worthy of some attention He would make very important discoveries in it. The greatest advantage would show itself, however, when one could find proofs for these theorems.

This passage is vintage Euler in that the basic idea is an insight so profound that it is crucial to much of algebraic number theory, yet at the same time many of the individual statements are patently false. The last statement of Theorem IV, for example, is clearly wrong. Not only is it not true that *all* prime numbers of the form $20n+3$ are of the form $5x^2+y^2$, but *no* prime numbers $20n+3$ are $5x^2+y^2$. To prove this it suffices to note that, since p is to be odd, x and y must have opposite parity, that is, either $x=2j+1$, $y=2k$ or $x=2c$, $y=2d+1$. In the first case

$$5x^2+y^2=(4+1)(4j^2+4j+1)+4k^2=4(4j^2+4j+1+j^2+j+k^2)+1$$

and in the second case

$$5x^2+y^2=4(5c^2+d^2+d)+1,$$

so in either case p is 1 more than a multiple of 4 and cannot have the form $4n+3$, much less the form $20n+3$, or the form $20n+4+3$.

Fortunately, the letter to Goldbach is only the first of many passages in his known writings where Euler deals with this subject, and in later versions the obvious mistakes are corrected. For example, in his main exposition **[2]** of these ideas he corrects the second part of Theorem IV to say that if p is a prime of the form $p=20n+1$ or $20n+9$ then $p=5x^2+y^2$, and if it is a prime of the form $20n+3$ or $20n+7$ then $2p=5x^2+y^2$. (Examples: $2\cdot 3=5\cdot 1^2+1^2$, $2\cdot 7=5\cdot 1^2+3^2$, $2\cdot 23=5\cdot 3^2+1^2$, $2\cdot 43=5\cdot 1^2+9^2$, $2\cdot 47=5\cdot 3^2+7^2$.) As restated, the theorem is correct and definitely not easy to prove.

The style of the corrected exposition **[2]** is similar to the letter above in that Euler first states a number of special theorems—covering the prime divisors of a^2+Nb^2 (a,b relatively prime) for $N=1,2,3,5,7,11,13,17,19,6,10,14,15,21,35,30$—before he states general theorems. This style has the advantage that the reader, far from having to struggle with the meaning of the general theorem, has probably become impatient with the special cases and has already made considerable progress toward guessing what the general theorem will be. Such a style is not appropriate to the sort of short note I am writing, however, and I will skip to the general case. Moreover, I will state it much more succinctly than Euler does.

THEOREM. *Let N be a given positive integer. Then there is a list $s_1, s_2, \dots, s_m$ of positive integers less than $4N$ and relatively prime to $4N$ with the following properties*:

(1) *Any odd prime number p which divides a number of the form a^2+Nb^2 without dividing either a or Nb is of the form $p=4Nn+s_i$ for some s_i in the list.*

(2) *Every prime number of the form $p=4Nn+s_i$ for some s_i in the list divides a number of the form a^2+Nb^2 without dividing either a or Nb.*

(3) *If s_i and s_j are in the list and if $s_is_j=4Nn+s$, $0<s<4N$, then s is in the list.*

(4) *If x is any integer less than $4N$ and relatively prime to $4N$ then either x or $4N-x$, but not both, are in the list.*

For example, when $N=13$, the list contains the 12 numbers 1, 7, 49, 31, 9, 11, 25, 19, 29, 47, 17, 15, that Euler gave in his letter to Goldbach. Property (4) becomes clearer if one writes $-x$ in place of $4N-x$ when $2N<4N-x<4N$ and reorders the list in order of the size of the absolute values of the entries. In the case $N=13$ this gives 1, -3, -5, 7, 9, 11, 15, 17, 19, -21, -23, 25, and in the general case it gives (by (4)) a list of the positive integers x less than $2N$ and relatively prime to $2N$ with a sign assigned to each. To see that property (3) holds in the case $N=13$ it suffices to note that Euler, in the letter, *derived* his list $1,7,49,31,\dots$ by repeatedly multiplying by 7 and removing multiples of 52. Thus, in the case $N=13$, the numbers s_i in the list are determined by $7^i=52n_i+s_i$ for $i=0,1,\dots,11$, and $7^{12}=52n_{12}+1$, from which (3) follows. Here are the lists described in the Theorem for a few values of N (see TABLE 1).

N	list											
1	1											
2	1,	3										
3	1,	-5										
4	1,	-3,	5,	-7								
5	1,	3,	7,	9								
6	1,	5,	7,	11								
7	1,	-3,	-5,	9,	11,	-13						
8	1,	3,	-5,	-7,	9,	11,	-13,	-15				
9	1,	5,	-7,	-11,	13,	17						
10	1,	-3,	7,	9,	11,	13,	-17,	19				
11	1,	3,	5,	-7,	9,	-13,	15,	-17,	-19,	-21		
12	1,	-5,	7,	-11,	13,	-17,	19,	-23				
13	1,	-3,	-5,	7,	9,	11,	15,	17,	19,	-21,	-23,	25

TABLE 1

I have included $N = 4, 8, 9, 12$ just to show that the Theorem applies in these cases, but Euler omits them for the simple reason that if you have the list for any N then you can trivially derive from it the list for Nk^2 for any k. For if p divides $a^2 + Nk^2b^2$ without dividing either a or Nk^2b then it divides $a^2 + N(kb)^2$ without dividing either a or Nkb, and on the other hand, if it divides $a^2 + Nb^2$ *and* if it does not divide k then it divides $(ka)^2 + Nk^2b^2$ without dividing either ka or Nk^2b.

A modern reader, after he sees the word THEOREM, expects to find the word *Proof* soon thereafter. However, customs were different in Euler's day and his paper contains 59 theorems without a single proof. He told Goldbach in his letter that "I have no demonstration of it as yet," and the fact is that he never found a demonstration of it or even of a substantial portion of it. His "theorems" were based on nothing but empirical evidence.

In order to test the Theorem empirically one needs to be able to test, given a prime number p and a positive integer N not divisible by p, whether there exist integers a and b not divisible by p such that p divides $a^2 + Nb^2$. This at first looks impossible to test because it looks like one must test an infinite number of values of a and b. However, a moment's reflection shows that one need only test values of a and b that are positive and less than p, because p divides $a^2 + Nb^2$ if and only if it divides $(a+p)^2 + Nb^2$ and the same holds for $a^2 + N(b+p)^2$, so multiples of p can be removed from a and b.

Using this observation, we can illustrate how one can test the Theorem, for example, for $N = 30$. Some numbers of the form $a^2 + 30b^2$ are

$$31,\ 34 = 2 \cdot 17,\ 39 = 3 \cdot 13,\ 46 = 2 \cdot 23,\ 55 = 5 \cdot 11,\ 66 = 2 \cdot 3 \cdot 11,\ 79, \text{and } 94 = 2 \cdot 47.$$

Thus the list must contain 31, 17, 13, 23, 11, $79 \equiv -41$, 47, where $\equiv$ indicates that 79 appears in the list as -41 when multiples of $4N = 120$ are removed to put the number between -60 and 60. More entries in the list can be found by using products of these. For example, $31 \cdot 17 = 527 \equiv 47$ is already in the list,

$$31 \cdot 13 = 403 \equiv 43,\ 31 \cdot 23 = 713 \equiv -7,\ 31 \cdot 11 = 341 \equiv -19,$$
$$31 \cdot (-41) = -1271 \equiv 49, \text{and } 31 \cdot 47 = 1457 \equiv 17.$$

A check shows that this assigns a sign to each positive integer less than 60 and relatively prime to 60 other than 1, 29, 37, 53, and 59. These are resolved by

$$17 \cdot 11 = 187 \equiv -53,\ 13 \cdot 23 = 299 \equiv 59,\ 23 \cdot 47 = 1081 \equiv 1,$$
$$11 \cdot (-41) = -451 \equiv 29, \text{and } 31 \cdot (-53) = -1643 \equiv 37.$$

Thus the list for $N = 30$ is

$$1,\ -7, 11, 13, 17,\ -19, 23, 29, 31, 37,\ -41, 43, 47, 49,\ -53, 59.$$

For any prime p, the Theorem now gives a prediction as to whether p does or does not divide a number of the form $a^2 + 30b^2$ without dividing a or $30b$, and this prediction can be checked in a finite number of steps. For example, it predicts that 37 *does* divide a number of this form, and, indeed, $9^2 + 30 = 111 = 3 \cdot 37$. It predicts that 7 *does not* divide a number of this form, and, indeed, a check of the 36 numbers $a^2 + 30b^2$, $0 < a < 7$, $0 < b < 7$, shows that none of them is divisible by 7. It is a long test to determine in this straightforward way whether a given p divides $a^2 + Nb^2$. The work can be greatly reduced by showing that if p divides any number of this type without dividing b then it divides a number of this type in which $b = 1$ and $0 < a < p/2$.* Thus in the case $p = 7$, $N = 30$, one need only check that 7 does not divide 31, 34, or 39 in order to conclude that the prediction of the Theorem is correct. Similarly, since 19 does not divide 31, 34, 39, 46, 55, 66, 79, 94, 111, the prediction for 19 is correct.

*Here is the argument. Since p does not divide b and p is prime, 1 is the greatest common divisor of p and b. The Euclidean algorithm can therefore be used to write $1 = Ap + Bb$ for integers A and B. If p divides $a^2 + Nb^2$ then it also divides $B^2a^2 + NB^2b^2 = c^2 + N(1 - Ap)^2$ and therefore divides $c^2 + N$. Now $c = qp + r$ where the remainder r can be taken in the range $-p/2 < r < p/2$ and p divides $r^2 + N$, as was to be shown.

In a few hours one could verify in this way the prediction of the Theorem in thousands of cases for dozens of values of N. Because the Theorem is so simple and general and withstands these tests so easily, one readily becomes convinced that it is true. Certainly Euler was convinced, so much so that at times he seems to have forgotten that the Theorem was completely unproved.

For simplicity, the case of negative N, that is, of prime divisors of $x^2 - Dy^2$ where $D > 0$, was omitted from the statement of the Theorem. It is easy to see that if D is a square then every prime p divides a number of this form. (For if $D = k^2$ then $x = k + p$ gives $x^2 - k^2 = p(2k + p)$, and p divides x only if it divides D.) However, if D is not a square then, as Euler already observed in his letter to Goldbach, a similar Theorem holds, except that instead of *never* containing both x and $-x$ the list in these cases *always* contains both whenever it contains either.

THEOREM (continued). *If N is negative and not of the form $-k^2$ then there is a list of integers s in the range $0 < s < |4N|$ and relatively prime to $4N$ such that* (1), (2), *and* (3) *hold* (*with $s < 4N$ changed to $s < |4N|$ in* (3)). *In this case* (4) *is replaced by*

(4′) *Exactly half the positive integers less than $|2N|$ and relatively prime to $2N$ are in the list, and x is in the list if and only if $|4N| - x$ is in the list.*

For example, here are the lists for a few negative values of N written, as before, with $-x$ in place of $|4N| - x$. The first three are from Euler's letter (see TABLE 2).

N	list					
-2	± 1					
-3	± 1					
-5	± 1,	± 9				
-6	± 1,	± 5				
-7	± 1,	± 3,	± 9			
-10	± 1,	± 3,	± 9,	± 13		
-11	± 1,	± 5,	± 7,	± 9,	± 19	
-13	± 1,	± 3,	± 9,	± 17,	± 23,	± 25

TABLE 2

Actually, there is a simple relation between the lists for N and $-N$ which can be summarized by saying that a number x of the form $4n + 1$ is either in *both* lists or it is in *neither*. For example, for $N = 7$, the numbers $1, 9, -3$ are in both lists and $5, 13, -11$ are in neither. It is possible in this way to find either list once the other is known. The relation is simple to prove* and it was well known to Euler.

It would be difficult to exaggerate the importance of this Theorem in the history of number theory. The effort to prove it surely spurred much of Euler's own later work, and the other two great number theorists of the 18th century, Lagrange and Legendre, also worked on topics around and about the Theorem without penetrating the Theorem itself. Finally, the young Gauss found a proof in 1796, and published two proofs in his great work, the *Disquisitiones Arithmeticae* in 1801. Gauss claimed to have discovered the Theorem on his own, but he would have needed to be in a cocoon in order not to have had *some* contact with work in this direction by Euler, Lagrange, and Legendre in the preceding half-century. I believe that Gauss was not being dishonest, but that he may have forgotten many subtle influences.

Gauss's formulation of the Theorem was very different from Euler's. For Euler, the basic question was whether, given N and p, the prime p divides a number of the form $x^2 + N$. It was noted above that if one can answer this question for N then one can easily deduce the answer for $-N$. A similar argument shows that if N is a product of two numbers $N = mn$ and if the question

*If $p = 4n + 1$ then, by the case $N = 1$ of the Theorem (which is one of the few cases that Euler later succeeded in proving) p divides $y^2 + 1$ for some y. If p also divides $x^2 + N$ for some x not divisible by p—i.e., if p is in the list for N—then p divides $x^2y^2 + Ny^2 = (xy)^2 - N + N(y^2 + 1)$, which shows that p divides $(xy)^2 - N$ and therefore that p is in the list for $-N$. Since N is not assumed to be positive in this argument, the same argument shows that if p is in the list for $-N$ it is also in the list for N.

can be answered for each factor m, n then it can be answered for N. (This becomes clear when the question "Is p in the list for N?" is restated "Is $-N$ a square mod p?" as below. If the answer is known for m and $-n$ then it is known for $N = mn$ because a product is a square if and only if *both* factors are squares or *neither* factor is a square.) Thus it suffices to be able to answer the question for $N = 1$ and N a prime. The cases $N = 1$ and $N = 2$ were resolved by Euler and Lagrange, so the question was reduced to the case where N is an odd prime. Thus the problem is in essence to find the list in Euler's Theorem when $\pm N$ is an odd prime. One can find this list without testing a single prime divisor of $x^2 \pm N$ if one observes that *the numbers common to the lists for N and $-N$, when N is prime, are precisely those numbers s, $-2N < s < 2N$, that can be written in the form $s = t^2 - 4Nk$ where t is a positive odd integer less than N.* This is a simple consequence of the fact that squares are necessarily in the list.*

For example, when $N = 11$, the numbers common to the lists are $1^2 = 1$, $3^2 = 9$, $5^2 \equiv -19$, $7^2 \equiv 5$, $9^2 \equiv -7$; thus the list for -11 is ± 1, ± 9, ± 19, ± 5, ± 7, and the list for 11 is 1, 3, 5, -7, 9, -13, 15, -17, -19, -21. When $N = 13$ the numbers in common are $1^2 = 1$, $3^2 = 9$, $5^2 = 25$, $7^2 \equiv -3$, $9^2 \equiv -23$, $11^2 \equiv 17$ so the lists are ± 1, ± 9, ± 25, ± 3, ± 23, ± 17 and 1, -3, -5, 7, 9, 11, 15, 17, 19, -21, -23, 25.

Gauss approached the subject from a different point of view, asking, for distinct odd primes p and q, whether q is a square mod p, that is, whether there is an integer x such that $x^2 - q$ is divisible by p. His "fundamental theorem," now known as the **law of quadratic reciprocity** because it describes a reciprocal relationship between the questions "Is q a square mod p?" and "Is p a square mod q?" states:

If p is of the form $p = 4n + 1$ then q is a square mod *p if and only if p is a square* mod *q.*
If p is of the form $p = 4n - 1$ then q is a square mod *p if and only if $-p$ is a square* mod *q.*

This is easy to deduce from the Theorem above**, easy enough that it is not stretching matters very far to say that the law of quadratic reciprocity is a consequence of Euler's theorems. However, for reasons to be explained in a moment, it is not in Euler's interest to stretch matters at all.

The law of quadratic reciprocity is the crowning theorem of elementary number theory. One might almost say that it is the theorem with which elementary number theory ceases to be elementary. Gauss, who did not waste time with trivialities, was fascinated by this theorem, so simple to state and so difficult to prove, and he returned to it many times in his career, giving six different proofs of it.

Gauss also studied *higher* reciprocity laws, which deal, roughly speaking, with the prime divisors of $x^3 - N$ (cubic reciprocity), $x^4 - N$ (biquadratic reciprocity), etc. The study of higher reciprocity laws was unquestionably the central question of 19th century number theory, engaging

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*To see this, note that if N is an odd prime then each list has $N - 1$ entries and half that many are common to the two lists. Therefore one need only show that all squares (reduced by subtracting multiples of $4N$ to put them between $-2N$ and $2N$) are in both lists, because this would account for all $(N - 1)/2$ common entries. For any of the $2N - 2$ nonzero odd integers x between $-2N$ and $2N$, multiplication by x and reduction by removing multiples of $4N$ is a one-to-one map of this set with $2N - 2$ elements to itself. For either of the two lists, if x is in the list then, by (3), multiplication by x carries elements of the list to elements of the list. Therefore, by counting, it carries elements not in the list to elements not in the list. In other words, if x is in the list and y is not then the reduction of xy is not in the list. Therefore multiplication by y carries elements of the list to elements not in the list. Since the list and its complement both have $N - 1$ elements, multiplication by y and reduction carries elements in the list one-to-one onto elements not in the list. By counting, then, it carries elements not in the list to elements of the list. Therefore if y is not in the list, the reduction of y^2 is. Thus the reduction of y^2 is in the list whether or not y is.

**Here is the argument. If $p = 4n + 1$ and p is a square mod q, say $p - z^2$ is divisible by q, then $y = z$ or $z + q$ is odd and $p - y^2$ is divisible by both 4 and q. Therefore p is in the list for $N = -q$ (and also for $N = q$), which means that $x^2 - q$ is divisible by p for some x, that is, q is a square mod p. Conversely, if q is a square mod p then p is in the list for $N = -q$. Therefore, since $p = 4n + 1$, p is in both lists and $p = t^2 - 4qk$, which shows that p is a square mod q. The proof in the case $p = 4n - 1$ is the same with p replaced by $-p$.

the best efforts of Jacobi, Eisenstein, Kummer, Hilbert, and many others, and leading to the creation of algebraic number theory. Two developments in the subsequent history of the subject give further testimony to Euler's genius and the importance of the theorems that he first announced to Goldbach.

First, a manuscript of Euler published in 1849 (he had died in 1783) showed that Gauss was not in fact the first to study higher reciprocity laws, but that Euler had already made some substantial progress on cubic reciprocity as early as 1749, and had not published his "theorems" in this field. For example, he stated the following conjecture:

> Let p be a prime of the form $3n+1$. Then 5 is a cube $\bmod p$ if and only if the representation of p in the form $p = x^2 + 3y^2$ satisfies one of the 4 conditions (1) $y = 15m$, (2) $x = 5k, y = 3m$, (3) $x \pm y = 15m$, or (4) $2x \pm y = 15m$.

(Theorem III of the letter to Goldbach may or may not assert the existence of such a representation $p = x^2 + 3y^2$ whenever $p = 3n + 1$, depending on one's interpretation of the phrase "contained in the form $3xx + yy$." In any case, Euler later not only asserted the existence of such a representation, he proved it rigorously.) Euler gave no indication of how he arrived at this astounding set of conditions, and the fact that they are correct struck the editor of the relevant volume of his collected works (Vol. 5 of the first series) as "bordering on the incomprehensible." However, the conjecture can be derived by applying the ideas described above to "imaginary primes" of the form $x + y\sqrt{-3}$ and finding the classes of imaginary primes $\bmod 3 \cdot 5$ for which 5 is a cube.

The second testimony to Euler's genius in the history of the subject is that later research showed that the "reciprocity law" approach to the subject was something of a blind alley. Hilbert in the 1890's formulated the quadratic and higher laws in terms of a simple product formula which was generally regarded as a more natural way of describing the basic phenomenon, and in which there is no "reciprocity" but, rather, an explicit formula for determining (in the quadratic cases) which classes $\bmod 4N$ contain prime divisors of $x^2 + Ny^2$. Later, in the 1920's, the subject reached what is generally regarded as its culmination in the form of the **Artin Reciprocity Law**, which, again, has no element of "reciprocity" in it. Moreover, in the quadratic case, Artin's Law is almost exactly the Theorem we have stated, which was discovered by Euler nearly 200 years earlier.

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[3] A. P. Juškevič and E. Winter, eds., Leonhard Euler und Christian Goldbach, Briefwechsel 1729–1764, Akademie-Verlag, Berlin, 1965.

Swiss banknote shows engraving of Euler (after Handmann's portrait, p. 261) on a background of Euler diagrams (see pp. 317–318) and interlocking gears.

Some Remarks and Problems in Number Theory Related to the Work of Euler

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Motto. *One is mathematics and Euler is its prophet.* This phrase was coined half as a joke at a mathematical party in Budapest about 50 years ago by Tibor Gallai. In these remarks we mention some of the things the prophet Euler has handed down to us and sometimes give some later developments. Many of the recollections and conjectures in these remarks are those of the first author, and first person references are used to keep the exposition informal.

In 1737 Euler proved that the number of primes was infinite by showing that the sum of their reciprocals diverges, i.e.,

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty. \tag{1}$$

He did this by using the (invalid) identity

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_p \left(1 - \frac{1}{p}\right)^{-1}$$

Though invalid—Euler rarely worried about convergence—it can be fixed by looking at

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

as $s \to 1$. For this, see Ayoub **[1]**, who said elsewhere **[2]** that Euler "laid the foundations of analytic number theory."

Denote by $\pi(x)$ the **number of primes** $p \leqslant x$. It is curious that Euler after having proved (1) never asked himself: how does $\pi(x)$ behave for large x? For (1) immediately implies that for infinitely many x, $\pi(x) > x^{1-\varepsilon}$. In fact, for infinitely many x, $\pi(x) > x/(\log x)^{1+\varepsilon}$. It seems to me that with a little experimentation Euler could have discovered the prime number theorem

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1.$$

After all, he did discover the quadratic reciprocity theorem by observation, and that seems to be at least as hard to see. But as we will see again later, such questions did not seem to occur to Euler. The prime number theorem was first conjectured shortly before Euler's death by Legendre in 1780 in the form

$$\pi(x) \approx \frac{x}{\log x - c},$$

with $c \approx 1.08$. In 1792 Gauss, who was only 15 at the time, even noticed that

$$\int_2^x \frac{dy}{\log y} = \sum_{k=2}^{x} \frac{1}{\log k} + O(1)$$

gives a much better approximation to $\pi(x)$ than $x/\log x$, a most remarkable achievement! Again, it is strange that Gauss and others did not prove that

$$\frac{c_1 x}{\log x} < \pi(x) < \frac{c_2 x}{\log x}$$

and that if $\lim_{x\to\infty} \pi(x)/(x/\log x)$ exists, then it must be 1. All these results were proved by Tchebychef around 1850. Both Euler and Gauss could easily have proved all this. The prime number theorem was first proved by Hadamard and de la Vallée Poussin in 1896 using analytic functions, which were not available to Euler and Gauss.

More than 40 years ago, I conjectured that if $1 \leqslant a_1 < a_2 < \cdots$ is a sequence of integers for which

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty, \tag{2}$$

then the sequence $\{a_n\}$ contains arbitrarily long arithmetic progressions. This conjecture is still not settled; I offer \$3,000 for its proof or disproof. If my conjecture is true, then Euler's result that $\Sigma 1/p$ diverges immediately implies that the primes contain arbitrarily long arithmetic progressions. Until this year, the longest such progression known was due to Weintraub [**33**] and has 17 terms: $3430751869 + 87297210t$, $t = 0, 1, \ldots, 16$. With patience and a good computer one could probably find more primes in arithmetic progression. In fact, 18 such primes were found by P. Pritchard, who reported this in January 1983 at the AMS meeting in Denver. The discovery was also described in *The Chronicle of Higher Education*, 2/9/83, p. 27.

It often happens that a problem on primes can be solved by generalizing it, and proving it for some more general sequences which share some property with the primes, such as being equally numerous. Even using this idea, my \$3,000 problem really seems to be very deep. Schur conjectured, and van der Waerden proved [**30**], that if we divide the integers into two classes, then at least one of the classes contains arbitrarily long arithmetic progressions. Fifty years ago, Turan and I conjectured [**13**] that if $r_k(n)$ is the smallest integer such that every sequence of integers of the form

$$1 \leqslant a_1 < a_2 < \cdots < a_{r_k(n)} \leqslant n$$

contains an arithmetic progression of k terms, then for every k,

$$\lim_{n\to\infty} \frac{r_k(n)}{n} = 0.$$

This conjecture is clearly stronger than van der Waerden's theorem, but weaker than (2). About 30 years ago K. F. Roth [**28**] proved the conjecture for $k = 3$. The general conjecture was finally proved by Szemeredi in 1972 [**29**]. For further information see [**14**].

A much stronger conjecture on primes states that for every k there are k *consecutive* primes which form an arithmetic progression. The longest known has only six terms: $121174811 + 30t$, $t = 0, 1, \ldots, 5$ [**19**]. This conjecture is undoubtedly true but is completely unattackable by the methods at our disposal.

Denote by $p(n)$ the **number of unrestricted partitions of** n, that is, the number of ways of writing n as a sum of positive integers. For example, $p(5) = 7$ because $1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 2 + 2 + 1 = 3 + 1 + 1 = 3 + 2 = 4 + 1 = 5$. Leibniz asked Bernoulli about $p(n)$ in 1669, but it was not until Euler saw that

$$1 + \sum_{n=1}^{\infty} p(n) x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$$

and ingeniously proved that

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n+1)/2}$$

that any progress was made. Combining the last two equations gives a recursion relation

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\cdots$$

that lets values like $p(200)=397299029388$ [15] be calculated. This was the start of generating functions.

As far as I know, Euler never tried to estimate $p(n)$ as a function of n. Hardy-Ramanujan [15] and Uspensky were the first to obtain the asymptotic formula for $p(n)$,

$$p(n)\sim\frac{1}{4\sqrt{3}\,n}e^{\pi\sqrt{2n/3}}. \tag{3}$$

In 1937 Rademacher [26] found a convergent series for $p(n)$ and later I [11] and Newman [24] gave an elementary proof of (3). These estimates are complicated, but the inequality

$$e^{c_1 n^{1/2}}<p(n)<e^{c_2 n^{1/2}}$$

could very easily have been obtained by Euler. These questions which seem so natural to us now must not have occurred to Euler. It could have been that the idea of function was not yet a natural one. Euler was more concerned with representing integers in various forms. He spent 40 years, off and on, trying to prove that every positive integer is a sum of four squares, only to have Legendre give the first proof in 1770. And think of how much time he must have spent on doing things like the following, finding integers x, y, z, w such that

$x\pm y,\ x\pm z,\ y\pm z$ are all squares (see below and right),

Tabelle

Für die Zahlen, welche in der Form m⁴—n⁴ enthalten sind.

m	nn	mm − nn	mm + nn	m⁴ − n⁴
4	1	3	5	3. 5
9	1	8	10	16. 5
9	4	5	13	5. 13
16	1	15	17	3. 5. 17
16	9	7	25	25. 7
25	1	24	26	16. 3. 13
25	9	16	34	16. 2. 17
49	1	48	50	25. 16. 2. 3
49	16	33	65	3. 5. 11. 13
64	1	63	65	9. 5. 7. 13
81	49	32	130	64. 5. 13
121	4	117	125	25. 9. 5. 13
121	9	112	130	16. 2. 5. 7. 13
121	49	72	170	144. 5. 17
144	25	119	169	169. 7. 17
169	1	168	170	16. 3. 5. 7. 17
169	81	88	250	25. 16. 5. 11
225	64	161	289	289. 7. 23

Euler's *Algebra* (v. 2, chap. 15, §235, p. 351) contains this table of squares m^2, n^2, their difference and sum, and m^4-n^4 (the left column heading has a printer's error).

$xy \pm x$, $xy \pm y$ are all squares,

$x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$ are all squares,

$x^2 + y^2 + z^2$, $x^2 + y^2 + w^2$, $x^2 + z^2 + w^2$, $y^2 + z^2 + w^2$ are all squares,

$x + y$ is a square, $x^2 + y^2$ is a cube,

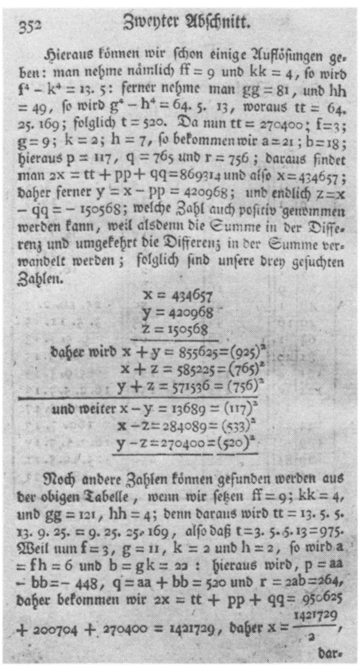
$x + y + z$, $xy + yz + zx$, xyz are all squares,

and so on ([8], ii, XV–XXI). Perhaps not many today are very interested in this.

Euler was the first to consider the function $\phi(n)$, the **number of integers $1 \leqslant m < n$ relatively prime to n**, and this function bears his name (see Glossary). Euler derived a formula equivalent to the well-known

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

but he never investigated the function any further, though a great deal of work has been done on it since. It is one more example of Euler's lack of curiosity about functions. There is still a surprising number of unsolved problems about $\phi(n)$. Carmichael conjectured [4] that the number of solutions of $\phi(n) = m$ can never be 1 (i.e., if $\phi(n_1) = m$ then there is an $n_2 \neq n_1$ with $\phi(n_2) = m$). Though the conjecture is known to be true for $m < 10^{400}$ [17], it is probably unattackable by the methods at our disposal. I proved [10] that if there is an integer m for which $\phi(n) = m$ has k solutions, then there are infinitely many integers with this property. If n is prime, $\phi(n) = n - 1$ of course; Lehmer conjectured [21] that $\phi(n)$ divides $(n-1)$ only if n is prime. This conjecture also seems unattackable. On the other hand, it is an easy exercise to show that $\phi(n)$ divides n if and only if $n = 2^\alpha 3^\beta$. R. L. Graham has conjectured that for every a there are infinitely many n for which $\phi(n)$ divides $n + a$.



352 Zweyter Abschnitt.

Hieraus können wir schon einige Auflösungen geben: man nehme nämlich ff = 9 und kk = 4, so wird $f^4 - k^4 = 13.\ 5$: ferner nehme man gg = 81, und hh = 49, so wird $g^4 - h^4 = 64.\ 5.\ 13$, woraus tt = 64. 25. 169; folglich t = 520. Da nun tt = 270400; f = 3; g = 9; k = 2; h = 7, so bekommen wir a = 21; b = 18; hieraus p = 117, q = 765 und r = 756; daraus findet man 2x = tt + pp + qq = 869314 und also x = 434657; daher ferner y = x − pp = 420968; und endlich z = x − qq = − 150568; welche Zahl auch positiv genommen werden kann, weil alsdenn die Summe in der Differenz und umgekehrt die Differenz in der Summe verwandelt werden; folglich sind unsere drey gesuchten Zahlen.

x = 434657
y = 420968
z = 150568

daher wird x + y = 855625 = $(925)^2$
x + z = 585225 = $(765)^2$
y + z = 571536 = $(756)^2$

und weiter x − y = 13689 = $(117)^2$
x − z = 284089 = $(533)^2$
y − z = 270400 = $(520)^2$

Noch andere Zahlen können gefunden werden aus der obigen Tabelle, wenn wir setzen ff = 9; kk = 4, und gg = 121, hh = 4; denn daraus wird tt = 13. 5. 5. 13. 9. 25. = 9. 25. 25. 169, also daß t = 3. 5. 5. 13 = 975. Weil nun f = 3, g = 11, k = 2 und h = 2, so wird a = fh = 6 und b = gk = 22: hieraus wird, p = aa − bb = − 448, q = aa + bb = 520 und r = 2ab = 264, daher bekommen wir 2x = tt + pp + qq = 950625 + 200704 + 270400 = 1421729, daher x = $\frac{1421729}{2}$,

dar-

Euler then demonstrates (p. 352), using his table values, how to obtain integers x, y, z such that sums and differences of any pair of these is a square. In his example, he obtains $x = 434{,}657$, $y = 420{,}968$, $z = 150{,}568$.

The well-known conjecture of Fermat states that $x_1^k + x_2^k = x_3^k$ has no positive integer solutions for $k > 2$. Euler proved the statement for $k = 4$ and almost proved it for $k = 3$ (see [**8**], ii, XXI, XXII). It has recently been proved by Wagstaff for all $k \leqslant 125{,}000$ [**31**]. The general conjecture seems to be out of reach at present. Euler conjectured the following generalization:

$$x_k^k = x_1^k + x_2^k + \cdots + x_{k-1}^k$$

has no nontrivial solution in integers for $k \geqslant 3$. This conjecture was disproved by Lander and Parkin [**18**] who found the equation

$$144^5 = 133^5 + 110^5 + 84^5 + 27^5.$$

This is so far the only known counterexample. The case $k = 4$ seems to be of special interest; in 1948 M. Ward [**32**] showed that there are no nontrivial solutions for $x_4 \leqslant 10{,}000$, and in 1967 Lander, Parkin and Selfridge [**20**] extended the result to $x_4 < 220{,}000$. Euler was not even able to find four fourth powers whose sum is a fourth power and it was only in 1911 that the example

$$353^4 = 315^4 + 272^4 + 120^4 + 30^4$$

was found by R. Norrie ([**8**]; ii, XXII).

In the same direction, Euler gave a complete parametric solution of the equation

$$x^3 + y^3 = u^3 + v^3,$$

namely,

$$\begin{aligned} x &= 1 - (a - 3b)(a^2 + 3b^2) \qquad & u &= (a + 3b) - (a^2 + 3b^2)^2 \\ y &= (a + 3b)(a^2 + 3b^2) - 1 \qquad & v &= (a^2 + 3b^2)^2 - (a - 3b) \end{aligned}$$

and proved that for infinitely many integers $n, n = x^4 + y^4 = u^4 + v^4$ by giving a complicated parametric solution [**16**] which includes the smallest solution

$$133^4 + 134^4 = 158^4 + 59^4 = 635,318,657.$$

After Ramanujan surprised Hardy by knowing that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3$$

was the smallest integer which is the sum of two cubes in more than one way, Hardy asked him if he knew any integer which was the sum of two fourth powers in more than one way. Ramanujan answered that he did not know any such numbers, and if they existed, they must be very large. Thus, both were unaware of the old work by Euler. It is not yet known if there are any integers which are the sum of two fourth powers in *more* than two ways, i.e., if the number of solutions of $n = x^4 + y^4$ is at most 2.

Denote by $f_3^{(2)}(n)$ the number of solutions of $n = x^3 + y^3$. Mordell proved that $\limsup_{n\to\infty} f_3^{(2)}(n) = \infty$ and Mahler [**23**] proved that $f_3^{(2)}(n) > (\log n)^{1/4}$ for infinitely many n. As far as I know there is no nontrivial upper bound known for $f_3^{(2)}(n)$. Very likely $f_3^{(2)}(n) < c_1(\log n)^{c_2}$ for all n, if c_1 and c_2 are sufficiently large absolute constants.

Euler was the first to evaluate $\sum_{n=1}^{\infty} 1/n^2$. In 1731 he obtained the sum accurate to 6 decimal places, in 1733 to 20, and in 1734 to infinitely many $(= \pi^2/6)$. Ayoub [**2**] said about his proof that "it opened up the theory of infinite products and partial fraction decomposition of transcendental functions and its importance goes far beyond the immediate application." Euler studied further what we now call the **Riemann ζ-function** $(= \sum_{n=1}^{\infty} n^{-s}$ when $\mathrm{Re}(s) > 1)$ and in 1749 he proved the functional equation

$$\zeta(1 - s) = \pi^{-s} 2^{1-s} \Gamma(s) \cos \frac{\pi s}{2} \zeta(s)$$

for $s = 1, 2, \ldots$ and said that he was certain it was true for all real s. It was not until 1859 that Riemann proved this.

As far as we know, Euler was the first to define **transcendental numbers** as numbers which are not the roots of algebraic equations. It is perhaps curious that he never proved their existence. The proof of Liouville was well within his reach. Maybe Euler considered the existence of transcendental numbers as self-evident, which by our standards, is certainly not the case.

Of course, not even Euler was perfect. His proofs of Fermat's Last Theorem for exponent 3, as well as his proof that every prime has a primitive root, are considered incomplete by our present standards. He regularly used infinite series without paying any attention to convergence (nevertheless his proofs are almost always correct except for rigor, which is easy to supply).

However, in at least one instance, Euler's intuition completely misled him and he produced a false "proof" which could not be corrected by methods at his disposal. Euler wanted to prove that $\sum_{n=1}^{\infty}\mu(n)/n=0$, where $\mu(1)=1$, $\mu(n)=0$ if n is not square-free and $\mu(n)=(-1)^k$ if n is the product of k distinct primes ($\mu(n)$ is known as the Möbius function). He simply argued as follows:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=\prod_{p}(1-1/p)=0. \tag{4}$$

This argument is, of course, inaccurate, since $\sum_{n=1}^{\infty}\mu(n)/n$ is not absolutely convergent and (4) was first proved correctly by von Mangoldt at the end of the nineteenth century. Equation (4) is known to be equivalent to the prime number theorem.

Another error of a different kind was pointed out to me by Mordell. Euler proved that if p divides x^2+ny^2 without dividing both x and y, then p is u^2+nv^2 for $n=1,2,3$. He then used the same arguments for $n=5$, though Fermat knew long before, and Euler knew too, that the conclusion was not true. (We know now that the reason is that unique factorization fails.) Edwards [**9**] thinks it was Euler's age, his blindness, or his secretary that may have caused the mistake.

We close with some of the less important things Euler did, to give an idea of his immense range and power. Before Euler, only three pairs of **amicable numbers** were known. These are pairs like 220 and 284, where the sum of the proper divisors of one of the numbers is equal to the other: $110+55+44+22+20+11+10+5+4+2+1=284$ and $142+71+4+2+1=220$. The pair (220, 284) was known to Pythagoras; another pair, (17296, 18416), was found by Fermat in 1636; and the pair (9363584, 9437056) was found by Descartes in 1638. In 1750, Euler gave *62* new pairs ([**8**], i, I). Amicable pairs are still studied. There were 1095 pairs known in 1972 [**22**] and a 152-digit pair was found in 1974 [**27**]. In 1955 I showed [**12**] that if $A(x)$ is the number of amicable pairs (m,n) with $m\leqslant n\leqslant x$, then $\lim_{x\to\infty}A(x)/x=0$; Pomerance showed in 1981 [**25**] that $A(x)<xe^{-(\log x)^{1/3}}$. In the other direction, I conjecture that there are infinitely many pairs; in fact, it is likely that $A(x)>cx^{1-\epsilon}$.

In a letter, Goldbach called Euler's attention to **multigrades**: sets of integers with equal sums of different powers, as in

$$1^k+5^k+9^k+17^k+18^k=2^k+3^k+11^k+15^k+19^k$$

for $k=0,1,2,3,4$, and Euler proved the first theorems about them. They have been studied a great deal since then. It was also in a letter to Euler that Goldbach made his famous conjecture that every even integer greater than 4 is a sum of two primes, and that has been studied even more than multigrades. There has not been much progress since Chen showed in 1966 [**5**] that every sufficiently large even integer is a sum of a prime and a product of at most two primes.

Euler discovered that if $p=4k+1$ is a prime, then p can be written $p=x^2+y^2$ in exactly one way; this led him to look for numbers d such that if $n=x^2+dy^2$ with $(x,y)=1$ in exactly one way, then n is prime. He found 65 of them, with 1848 the largest ([**8**], ii, XIV). It seems likely that he found them all, since it is known that their number is finite [**6**] and there is at most one greater than 10^{65} [**7**]. So in a way, Euler said the first and last words on this subject.

Euler proved that every **even perfect number** (i.e., equal to the sum of its proper divisors, as $28=14+7+4+2+1$) is of the form $2^{p-1}(2^p-1)$ for p and 2^p-1 prime and gave the first of a long list of necessary conditions that an odd perfect number will have to satisfy ([**8**], i, I).

Fermat thought all the Fermat numbers $2^{2^n}+1$ were prime. Euler factored $2^{2^5}+1$ in 1732; $2^{2^7}+1$ was not factored until 1971 [**3**].

Euler was the first to look at that equation that keeps coming up in popular journals, $x^y = y^x$ ([**8**], ii, XXIII).

And Euler discovered, no one knows how, that the polynomial $n^2 - n + 41$ is a prime for $n = 1, 2, \ldots, 40$.

If Euler had never done anything *except* number theory, he would still be remembered as one of the great mathematicians.

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Euler's Vision of a General Partial Differential Calculus for a Generalized Kind of Function

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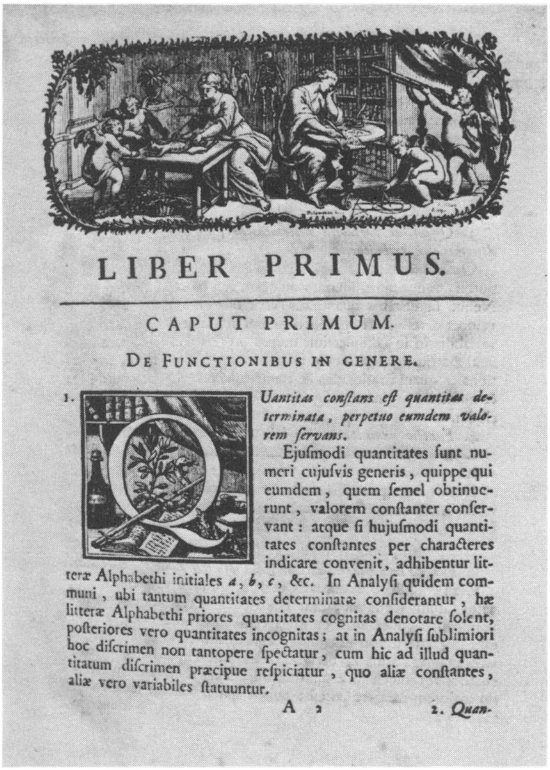
The vibrating string controversy involved most of the analysts of the latter half of the 18th century. The dispute concerned the type of functions which could be allowed in analysis, particularly in the new partial differential calculus. Leonhard Euler held the bold opinion that all functions describing any curve, however irregular, ought to be admitted in analysis. He often stressed the importance of such an extended calculus, but did almost nothing to support his point of view mathematically. After having been abandoned during the introduction of rigor in the latter part of the 19th century, Euler's ideas began to take more concrete form during the early part of the 20th century, and they have now been incorporated into L. Schwartz's theory of distributions.

The algebraic function concept

Euler's radical stand in the dispute over the vibrating string is surprising since he had canonized the narrower range of analysis which his main opponent, J. B. R. d'Alembert (1717–1783), adhered to. This was done in the influential book *Introductio in analysin infinitorum*



LIBER PRIMUS.

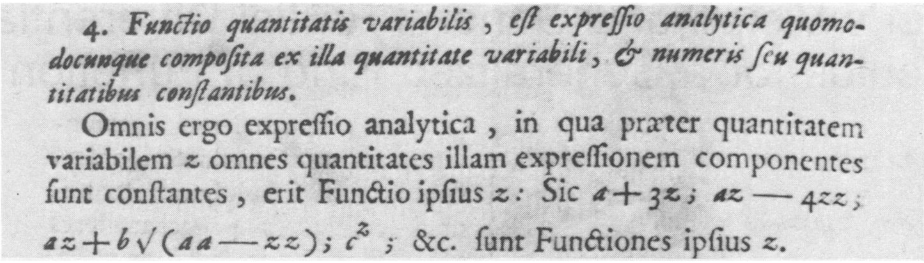
CAPUT PRIMUM.

DE FUNCTIONIBUS IN GENERE.

1. *Quantitas constans est quantitas determinata, perpetuo eumdem valorem servans.*

Ejusmodi quantitates sunt numeri cujusvis generis, quippe qui eumdem, quem semel obtinuerunt, valorem constanter conservant: atque si hujusmodi quantitates constantes per characteres indicare convenit, adhibentur litteræ Alphabethi initiales *a*, *b*, *c*, &c. In Analysi quidem communi, ubi tantum quantitates determinatæ considerantur, hæ litteræ Alphabethi priores quantitates cognitas denotare solent, posteriores vero quantitates incognitas; at in Analysi sublimiori hoc discrimen non tantopere spectatur, cum hic ad illud quantitatum discrimen præcipue respiciatur, quo aliæ constantes, aliæ vero variabiles statuuntur.

A 2   2. *Quan-*



4. *Functio quantitatis variabilis, eſt expreſſio analytica quomodocunque compoſita ex illa quantitate variabili, & numeris ſeu quantitatibus conſtantibus.*

Omnis ergo expreſſio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expreſſionem componentes ſunt conſtantes, erit Functio ipſius z: Sic $a+3z$; $az-4zz$; $az+b\sqrt{(aa-zz)}$; c^z; &c. ſunt Functiones ipſius z.

[12], in which Euler chose to determine the relation between the variable quantities by way of functions instead of using curves, as had been universally done earlier (cf. [22] and [7]). He defined a function as follows (see photo above):

> A function of a variable quantity is an analytical expression composed in one way or another of this variable quantity and numbers or constant quantities [12, ch. 1, § 4].

In forming the analytical expressions, Euler allowed the use of the standard transcendental operations such as log, exp, sin and cos in addition to algebraic operations. Still, all the rules in the theory of functions were taken over from algebra, so that Euler's function concept was in essence entirely algebraic. Thus *Introductio* marked a shift in the setting of analysis from geometry to algebra. Euler even accepted, and treated algebraically, infinite expressions such as infinite series, infinite products and continued fractions. Lebesgue [25] later showed that when such infinite limit procedures are accepted, the class of functions is very extensive, namely, equal to the class of Borel Functions. However, Euler did not realize the immense generality of his function concept and in theoretical considerations he conferred on them all the nice properties he needed such as differentiability and even analyticity in the modern sense. Still, it would be off the mark to identify Euler's functions with one of the modern classes of functions such as differentiable functions or analytical functions because their definition involves topological (geometrical) ideas which are foreign to Euler's way of thinking.

Most important among the nice properties shared by all Euler's functions was the possibility of expanding them in a power series:

$$f(x+i)=f(x)+pi+qi^2+ri^3+\cdots,$$

for in all differentiations actually carried out in Euler's second influential textbook *Institutiones calculi differentialis* [16] the differential quotient is found as the coefficient p of the first power term. Later in the century J. L. Lagrange [24] *defined* the derivative of a function in this way and gave a "proof" that the expansion always exists. In the mid-18th century, however, power series were only used as a practical tool whereas the metaphysical basis for the calculus was found elsewhere. For example, d'Alembert defined the derivative using limits, and Euler's definition of the differential rested on a theory of zeros of different order. Yet, these foundational differences were not reflected in the domain they assigned to the ordinary calculus; both agreed that

> "...[calculus] as it has been treated until now can only be applied to curves, whose nature can be contained in one analytical equation" [18, § 7].

Euler's generalized functions

The discussion of the vibrating string brought an end to this agreement. D'Alembert, who in 1747 [1] found his famous solution

$$y=f(x,t)=\phi(x+t)+\psi(x-t)$$

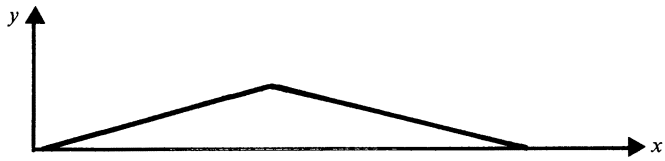


FIGURE 1

of the wave equation

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial t^2}$$

governing the displacement y of the string, required that the "arbitrary" functions ϕ and ψ be analytical expressions.

> In all other cases the problem cannot be solved, at least not with my method, and I do not even know whether it will not be beyond the powers of the known analysis. In fact, it seems to me that one cannot express y analytically in a more general way than supposing it to be a function of x and t [**2**, p. 358].

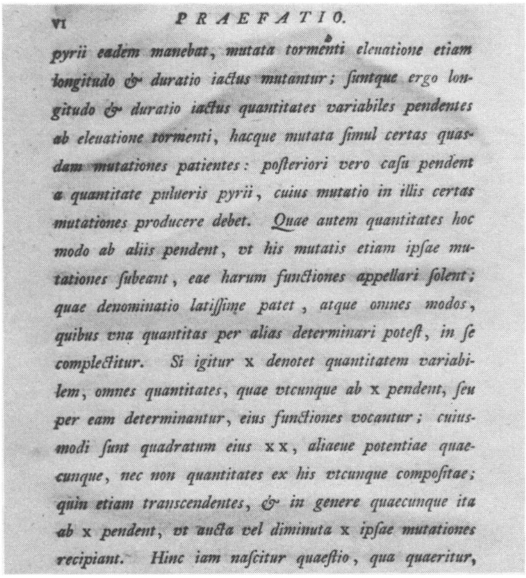
Euler, on the other hand, pointed out that this requirement restricted the initial displacement $\phi(x)+\psi(x)$ of the string too much; for example, he believed that the plucked string (FIGURE 1) would be excluded from d'Alembert's solution. (However, the plucked string can be described analytically by a slight modification of Cauchy's example: $\sqrt{x^2}=|x|$ [**9**].) Therefore he argued that one had to allow the functions ϕ and ψ to represent arbitrarily given curves. In this way physical reality led Euler to generalize the function concept so as to be in one to one correspondence with the geometrical concept of curve which he had earlier abandoned as the basic concept in analysis.

It is surprising that Euler never provided a proper definition of the more general notion of function. His many papers on the vibrating string (particularly [**17**]) made clear that a generalized function was something corresponding to a general hand-drawn curve, but he never explicitly stated what this something was supposed to be. To judge from the classification of the new functions he seems to have had an algebraic definition in mind. He divided the general functions into the continuous and the discontinuous. The former were identical with the functions defined in *Introductio*, whereas the latter could not be expressed by one analytical expression. Euler was quite explicit about the continuity of a function having nothing to do with the connectedness of the curve; for example $1/x$ is continuous but its graph is disconnected at $x=0$. Thus Euler's concept of continuity must be distinguished from the modern concept, due to Cauchy [**8**], and so we shall term the former ***E*-continuity**. In [**17**] Euler further divided the E-discontinuous functions into mixed functions, whose graph can be represented piecewise by finitely many analytical expressions, and the functions corresponding to arbitrary hand-drawn curves, whose analytical expressions may, so to speak, change from point to point.

Thus Euler's division of functions into classes was entirely algebraic and so was his distinction between even and odd functions. For example, in his critique [**15**] of D. Bernoulli's [**6**] description of the vibrating string as a trigonometric series, Euler argued that an E-discontinuous function of the form

$$\begin{cases} f(x) & \text{for } x>0 \\ -f(-x) & \text{for } x<0 \end{cases}$$

is only odd if f is odd and by that he meant that its power series contains only odd powers of x. To conclude: even when the consequences were absurd, Euler continued to think algebraically about his new functions, which, implicitly, he defined as the collection of the (possibly infinitely many) analytical expressions describing the corresponding curve.



VI PRAEFATIO.

pyrii eadem manebat, mutata tormenti eleuatione etiam longitudo & duratio iactus mutantur; ſuntque ergo longitudo & duratio iactus quantitates variabiles pendentes ab eleuatione tormenti, hacque mutata ſimul certas quasdam mutationes patientes: poſteriori vero caſu pendent a quantitate pulueris pyrii, cuius mutatio in illis certas mutationes producere debet. Quae autem quantitates hoc modo ab aliis pendent, vt his mutatis etiam ipſae mutationes ſubeant, eae harum functiones appellari ſolent; quae denominatio latiſſime patet, atque omnes modos, quibus vna quantitas per alias determinari poteſt, in ſe complectitur. Si igitur x *denotet quantitatem variabilem, omnes quantitates, quae vtcunque ab* x *pendent, ſeu per eam determinantur, eius functiones vocantur; cuiusmodi ſunt quadratum eius* xx, *aliaeue potentiae quaecunque, nec non quantitates ex his vtcunque compoſitae; quin etiam transcendentes, & in genere quaecunque ita ab* x *pendent, vt aucta vel diminuta* x *ipſae mutationes recipiant. Hinc iam naſcitur quaeſtio, qua quaeritur,*

Strangely enough, Euler himself had introduced a way of thinking about functions which he could have used to define his E-discontinuous functions as separate entities. In his second textbook on analysis *Institutiones calculi differentialis* (1755) [**16**], he defined functions in the following way (see photo above):

> If, therefore, x denotes a variable quantity, all quantities which depend in some way on x or are determined by it, are called functions of this variable [**16**, Preface].

As it stands, this is almost the modern function definition and it clearly encompasses the E-discontinuous functions. However, Euler did not realize its generality. In *Institutiones calculi differentialis* only E-continuous functions occur, and the E-discontinuous functions are not even mentioned. Neither did he refer to his 1755 definition in any of his later papers on E-discontinuous functions. This indicates that Euler thought of his 1755 function definition as being equivalent to the definition given in *Introductio*. In fact, Euler's statement from 1765 (quoted p. 300) that analysis until then had exclusively been concerned with analytical expressions only makes sense under this assumption. (This point of view is different from the one put forward by Youschkevich [**34**].)

Euler's vision of a generalized calculus

The lack of a proper definition of the E-discontinuous functions suggests that Euler's main concern was not the foundation of the generalized function concept itself but the analysis it made possible. We saw that initially Euler had introduced his new functions for physical reasons. Later [**17**] he stressed that the E-discontinuous functions were not forced onto analysis from outside but inevitably emerged as arbitrary functions in the partial integral calculus. For example [**20**, book 2, sect. 1, § 33], the solution of the partial differential equation

$$\frac{\partial u(x,y)}{\partial x}=0$$

is an arbitrary constant under the variation of x, but the constant can vary as a function f of y. It does not matter whether the constants for different values of y are connected by an analytical expression or not; therefore f must be allowed to be E-discontinuous. Since the functions ϕ and ψ in the solution of the wave equation arise in this way when $x+t$ and $x-t$ are used as independent variables, these functions are by their nature general functions.

Euler only used the E-discontinuous functions in the calculus of functions of several variables, but within that theory he would apparently blaze the trail for their unrestricted application. In contrast to the conservative d'Alembert, Euler argued that the development of a calculus of E-discontinuous functions is particularly desirable because all earlier calculus had been restricted to analytic expressions:

> But if the theory [of the vibrating string] leads us to a solution so general that it extends to all discontinuous as well as continuous figures, one must admit that this research opens to us a new road in analysis by enabling us to apply the calculus to curves which are not subject to any law of continuity, and if that has appeared impossible until now the discovery is so much more important [**18**, § 8].

Euler's insistence that calculus should be applicable within the whole new function domain instead of being restricted to some—possibly varying—subclass(es) (as is the case in modern analysis) was supported not only by the mentioned physical reasons. It was also in agreement with the fundamental belief in the generality of mathematics. For algebraic rules were considered universally valid because they operated on abstract quantities, and since analysis was just infinite algebra, its rules had to be generally applicable as well.

> For, because this calculus applies to variable quantities, that is, quantities considered generally, if it were not generally true... one could never make use of this rule, since the truth of the differential calculus is based on the generality of the rules of which it consists [**14**, 1. Objection].

This basic belief in the generality of mathematics forced Euler to extend calculus to all E-discontinuous functions as soon as he had allowed them to enter his mathematical universe. Initially it probably also made him believe that this extension would come down to a simple admission of all the well-known rules to the extended domain. However, he soon had to realize that d'Alembert's exclusion of E-discontinuous functions was not only due to plain conservatism but was supported by mathematical arguments.

In many examples d'Alembert showed that the mathematical analysis of the vibrating string broke down at points where ϕ or ψ changed their analytical expression. For example, d'Alembert [**3**, § 7] proved that if ψ is composed of two symmetric parabolas as in FIGURE 2 and $\phi \equiv 0$ then $\psi(x-t)$ does not satisfy the wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$$

at points where $x - t = 0$. This and other difficulties can be explained in modern terminology by the fact that ϕ or ψ are not twice differentiable. D'Alembert came close to such an insight towards the end of his life [**4**], but while the controversy was at its highest, he believed that he had proved that ϕ and ψ must be E-continuous.

Euler was not convinced by d'Alembert's arguments and tried to refute them with a few counterarguments [**19**] of which I shall reproduce the most convincing. He remarked that the trouble was due to the sharp bend in the first derivative of ψ. Therefore, one had only to smooth

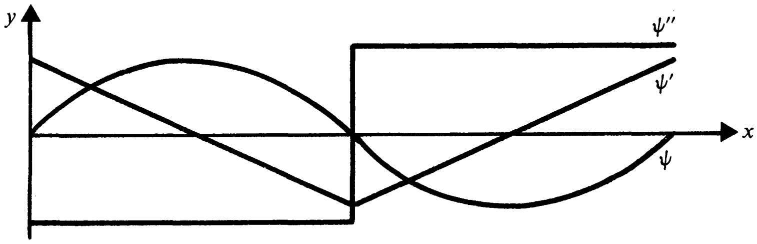


FIGURE 2

out ψ' which could be done by changing ψ infinitely little to $\tilde{\psi}$. Since $\tilde{\psi}(x-t)$ would then satisfy the wave equation, one also had to admit $\psi(x-t)$ as a solution since infinitely small changes were always ignored in analysis.

In Eulerian calculus this argument is not completely off the mark, and even in modern analysis it contains the germ of a good idea (cf. p. 305 of this article). Still Euler seems to have realized that he had not overcome all objections to his new general analysis, and so he often encouraged the younger mathematicians to work on these problems.

> This part of analysis [of two or more variables] is essentially different from the former [of one variable], and extends even to functions void of all law of continuity. This part, of which we so far know barely the first elements, certainly deserves the united efforts of all geometers for its investigation and development [**19**, § 32].

The fate of Euler's vision

In order to follow how subsequent geometers cultivated this new branch of analysis it is useful to divide the complex of problems, seen by Euler as a unity, into three separate parts:

(1) The generalization of the concept of function.
(2) The generalization of analysis.
(3) The development of the theory of partial differential equations.

The last and most important point of this research programme (3) was enthusiastically taken up by most of the mathematical community and was probably the most important mathematical discipline during the following half century. However, a discussion of it is far beyond the scope of this paper (see [**23**, ch. 22, 28]).

The generalization of the function concept (1) was also gradually accepted. In this process Euler's 1755 function definition was influential, regardless of his own interpretation of it. For after 1755 it became normal to reproduce this definition in textbooks on analysis, and slowly mathematicians began to realize its true generality. But this process took almost a century. For example, Lagrange [**24**] and Cauchy [**8**] defined functions generally as correspondences between variables, but they both thought of them as analytical expressions. It is natural in Lagrange's case, because he carried Euler's algebraic approach to its extreme, but it is surprising that the father of modern analysis, Cauchy, had a similar way of thinking. Still, this is evident from many remarks in his famous *Cours d'Analyse* [**8**], for example, the talk about "the constants or variables contained in a given function" [**8**, ch. 8, § 1].

In J. Fourier's works [**21**, § 417], one can find some comprehension of the generality of Euler's 1755 definition but the first mathematician who really took it seriously and understood the implications of the permissible pathologies was J. P. G. Lejeune-Dirichlet [**11**], after whom our function concept is justly named.

The generalization of analysis (2) suffered the opposite fate. At first it gained widespread acceptance but during the 19th century the idea was entirely abandoned. It happened as follows. In 1787 the St. Petersburg Academy officially terminated the controversy over the vibrating string by awarding L. Arbogast the first prize for a paper on the irregularities of arbitrary functions in the solutions of partial differential equations. Arbogast came out in favor of Euler's point of view, but he added nothing new to the foundational difficulties [**5**].

However, this official support of a general calculus was brushed aside by Cauchy, whose partial rigorization of analysis was a frontal attack on the principle of the generality of algebraic and analytical rules which had philosophically supported Euler's point of view. Cauchy explicitly pointed out this fundamental shift in the introduction to his famous *Cours d'Analyse* [**8**]:

> As for the methods, I have tried to give them all the rigour that one demands in geometry, so as never to have recourse to reasoning drawn from the generality of algebra.

Therefore nothing in his philosophy prevented him from confining calculus to a subclass of the class of functions, and in essence he restricted its use to the continuous functions (in the modern sense). In some of his papers he realized the inadequacy of this restriction, but a clear idea of the spaces $C^{(n)}(\mathbb{R})$ as the domain of d^n/dx^n did not crystalize until the 1870s in the Weierstrass school.

As a whole, mathematics benefited from this rigorization of analysis, but the corresponding restriction in the allowable solutions to partial differential equations made life complicated for the applied mathematician. Thus when irregular physical situations occurred (as, for example, a sharp bend in a string), the differential equation could not be used and a new mathematical model of the system had to be found. Such alternative models were set up, for example, by E. Christoffel [**10**].

However, in the beginning of the 20th century this procedure was felt to be so cumbersome and unnatural that several definitions of generalized solutions to partial differential equations were suggested, beginning in 1899 with H. Petrini's generalization of Poisson's equation [**28**]. Of the many generalization procedures I shall mention only the "sequence definition" implicitly used by N. Wiener in 1926 [**33**] and explicitly introduced by Sobolev (1935) [**32**]. According to this definition, f is a generalized solution to a (partial) differential equation if there exists a sequence of ordinary solutions $\{f_n\}$ converging, in a suitable topology, to f. This definition is particularly interesting because it leads to a sensible interpretation of Euler's argument against d'Alembert (pp. 303–304); for, if instead of one smooth function $\tilde{\psi}$ infinitely close to ψ, we think of a sequence ψ_n of such functions, then Euler's argument shows that $\psi(x-t)$ is a generalized solution to the wave equation.

All the ad hoc definitions of generalized solutions from the first half of this century were incorporated in the theory of distributions created by L. Schwartz during the period 1945–1950 [**31**] as a result of his work with generalized solutions to the polyharmonic equation [**30**]. The theory of distributions probably constitutes the closest approximation to Euler's vision of a general calculus one can obtain, for in that theory any generalized function is infinitely often differentiable. However, in many respects the reality has turned out to be different from the dream. In one respect the reality is more satisfactory since it not only generalizes partial differential calculus which Euler had imagined but encompasses ordinary differential calculus as well. In other respects it is less perfect; for example, the general use of the algebraic operations, such as multiplication of two generalized functions, has been sacrificed in the theory of distributions. Moreover, the necessary generalization of the function concept has turned out to be much more extensive than the one Euler suggested.

Concluding remarks

Surely the realization of Euler's vision of a general calculus was different from what he had imagined—and more difficult. This can only increase our admiration for his readiness to overthrow his own framework of analysis when physical reality called for it. His conduct reveals an undogmatic and flexible attitude toward the foundational problems, from which much could be learned by modern mathematicians. On the other hand, it is worth noting that the creation of the theory of distributions made extensive use of the classical theory of differential operators created more in the spirit of d'Alembert; one can even argue that the establishment of a secure foundation for the more restricted classical calculus was a necessary condition for the realization of Euler's vision of a general calculus.

As further reading on the development of the concept of function I can recommend [**34**], [**29**] and, for those who want to brush up their Danish, [**26**]. The book [**27**] contains more information on the history of generalized solutions to partial differential equations and other aspects of the prehistory of the theory of distributions.

The author and the editor appreciate the assistance of Dr. Dorothy Tyler in translating passages from Euler's works.

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Euler and Infinite Series

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The history of mathematics is valuable as an account of the gradual development of the many current branches of mathematics. It is extremely fascinating and instructive to study even the *false* steps made by the greatest minds and in this way reveal their often unsuccessful attempts to formulate correct concepts and proofs, even though they were on the threshold of success. Their efforts to justify their work, which we can now appraise with the advantage of hindsight, often border on the incredible.

These features of history are most conspicuous in the work of Leonhard Euler, the key figure in 18th-century mathematics, and one who should be ranked with Archimedes, Newton, and Gauss. Euler's recorded work on infinite series provides a prime example of the struggles, successes and failures which are an essential part of the creative life of almost all great mathematicians. The few examples discussed in this paper will serve to illustrate how Euler surmounted the difficulties he encountered.

Euler first undertook work on infinite series around 1730, and by that time, John Wallis, Isaac Newton, Gottfried Leibniz, Brook Taylor, and Colin Maclaurin had demonstrated the series calculation of the constants e and π and the use of infinite series to represent functions in order to integrate those that could not be treated in closed form. Hence it is understandable that Euler should have tackled the subject. Like his predecessors, Euler's work lacks rigor, is often *ad hoc*, and contains blunders, but despite this, his calculations reveal an uncanny ability to judge when his methods might lead to correct results. Our discussion will not follow the precise historical order of Euler's investigations of series; he made contributions throughout his lifetime.

To appreciate the first example of Euler's work on series, we must consider some background. A series which caused endless dispute was

$$1-1+1-1+\cdots. \tag{1}$$

It seemed clear that by writing this series as

$$(1-1)+(1-1)+(1-1)+\cdots$$

the sum should be 0. It seemed equally clear, however, that by writing the series as

$$1-(1-1)-(1-1)-\cdots$$

the sum should be 1. But still another sum seemed as reasonable. If S denotes the sum of the series (1), then

$$S=1-(1-1+1-1+\cdots)=1-S.$$

Hence $S=\frac{1}{2}$. This value was also supported by the formula for summing a geometric series with common ratio -1.

Guido Grandi (1671–1742), in his little book *Quadratura circula et hyperbolae per infinitas hyperbolas geometrice exhibita* (1703), obtained the third result by a variant of the geometric series argument, using the binomial expansion

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots,$$

with $x=1$. (He also argued that since the sum was both 0 and $\frac{1}{2}$, he had proved that the world could be created out of nothing.) In a letter to Christian Wolf published in the *Acta eruditorum* of 1713, Leibniz agreed with Grandi's result but thought that it should be possible to obtain it without resorting to the function $1/(1+x)$. He argued that, since the successive partial sums are

$1, 0, 1, 0, 1, \cdots$, with 1 and 0 equally probable, one should therefore take $\frac{1}{2}$, the arithmetic mean, as the sum. This argument was accepted by James, John and Daniel Bernoulli. Leibniz conceded that his argument was more metaphysical than mathematical, but said that there is more metaphysical truth in mathematics than is generally recognized.

Euler took a hand in this argument. To obtain the sum of the series (1), he argued in a manner similar to Grandi, substituting $x = -1$ in the expansion

$$1/(1-x) = 1 + x + x^2 + \cdots$$

and obtained

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots.$$

At this early stage of his work on series, Euler used expansion of functions into series to sum other divergent series. For example, he substituted $x = -1$ in the expansion

$$1/(1+x)^2 = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

and obtained

$$\infty = 1 + 2 + 3 + 4 + \cdots. \tag{2}$$

To Euler, this seemed reasonable; he treated infinity as a number. He then considered the geometric (or binomial) series for $1/(1-x)$ with $x = 2$ and obtained

$$-1 = 1 + 2 + 4 + 8 + \cdots. \tag{3}$$

Since the terms of series (3) exceed the corresponding terms of series (2), Euler concluded that the sum -1 is larger than infinity. Some of Euler's contemporaries argued that negative numbers larger than infinity are different from those less than 0. Euler objected and argued that infinity separates positive and negative numbers just as 0 does.

In a paper of 1734/35 [7], Euler started with the series

$$y = \sin x = x - x^3/3! + x^5/5! + \cdots \tag{4}$$

and rewrote the equation in the form

$$1 - x/y + x^3/3!y + x^5/5!y - \cdots = 0. \tag{5}$$

He then treated the left side of (5) as an infinite polynomial and argued as follows. (The argument is based on the fact that the sum of the reciprocals of the roots of the polynomial $p(x) = 1 - a_1x + a_2x^2 - a_3x^3 + \cdots + (-1)^k a_k x^k$ is a_1, the sum of the squares of the reciprocals of the roots of $p(x)$ is $a_1{}^2 - 2a_2$, and so on, for higher roots.) Let $A, B, C, \cdots$ be solutions of equation (5). Then the polynomial can be factored into an infinite product,

$$1 - \frac{x}{y} + \frac{x^3}{3!y} - \frac{x^5}{5!y} + \cdots = \left(1 - \frac{x}{A}\right)\left(1 - \frac{x}{B}\right)\left(1 - \frac{x}{C}\right)\cdots.$$

If A is the smallest value of x whose sine is y, then all other solutions $B, C, \cdots$ are $\pi - A, 2\pi + A, 3\pi - A, \cdots; -\pi - A, -2\pi + A, -3\pi - A, \cdots$. Thus

$$\frac{1}{A} + \frac{1}{\pi - A} + \frac{1}{2\pi + A} + \cdots - \frac{1}{\pi + A} - \frac{1}{2\pi - A} - \frac{1}{3\pi + A} - \cdots = \frac{1}{y}. \tag{6}$$

$$\frac{1}{A^2} + \frac{1}{(\pi - A)^2} + \frac{1}{(2\pi + A)^2} + \cdots + \frac{1}{(\pi + A)^2} + \frac{1}{(2\pi + A)^2} + \frac{1}{(3\pi + A)^2} + \cdots = \frac{1}{y^2} \tag{7}$$

and so on for higher powers of the reciprocals. If, in equations (4) and (5), we take $y = 1$, then $A = \pi/2$, so that (6) becomes

$$\frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = 1,$$

or

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

and (7) becomes

$$\frac{8}{\pi^2}\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots\right)=1,$$

or

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}. \tag{9}$$

Other series that were "summed" in the same manner are

$$1/1^3-1/3^3+1/5^3-\cdots=\frac{\pi^3}{32},$$

$$1/1^4+1/3^4+1/5^4+\cdots=\frac{\pi^4}{96},$$

$$1/1^5-1/3^5+1/5^5-\cdots=\frac{5\pi^5}{1536},$$

$$1/1^6+1/3^6+1/5^6+\cdots=\frac{\pi^6}{960},$$

and so on. From these series he deduced others. For example, since

$$\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)-\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)=\frac{1}{2^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right),$$

one can use (9) to obtain

$$1/1^2+1/2^2+1/3^2+1/4^2+\cdots=\frac{\pi^2/8}{3/4}=\frac{\pi^2}{6},$$

and in a similar manner, obtain

$$1/1^4+1/2^4+1/3^4+1/4^4+\cdots=\frac{\pi^4}{90}$$

and other sums. R. Ayoub [**1**] discusses Euler's use of (4) to compute such sums, W. F. Eberlein [**3**] discusses Euler's use of the infinite product for the sine function, and H. H. Goldstine [**9**, 3.1, 3.2] indicates Euler's expansions of such functions as $\frac{1}{2}(e^x-e^{-x})$ and the use of these expansions in computing sums such as (9).

Euler's attempts to sum the reciprocals of powers of the positive integers were not completely idle. In another paper of the same period [**4**], Euler made a somewhat bizarre use of infinitesimal calculus to find the difference between the sum of the harmonic series and the logarithm, a difference whose expansion utilizes precisely these series of powers. Let

$$s=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}.$$

If we regard n as infinite, then 1 is an infinitesimal and we can write $ds=1/n=1/n\,dn$. An integration yields

$$s=\log n+C.$$

To find C, note that

$$\frac{1}{x}=\log\left(1+\frac{1}{x}\right)+\frac{1}{2x^2}-\frac{1}{3x^3}+\frac{1}{4x^4}-\frac{1}{5x^5}+\cdots.$$

Setting $x=1,2,3,\ldots,n-1$, in turn, and adding the $n-1$ equations yields

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}=\log n+\frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n-1)^2}\right)$$
$$-\frac{1}{3}\left(1+\frac{1}{8}+\frac{1}{27}+\cdots+\frac{1}{(n-1)^3}\right)$$
$$+\frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{81}+\cdots+\frac{1}{(n-1)^4}\right)$$
$$-\cdots$$

The limiting value γ of C as n becomes infinite is today called **Euler's constant.**

In a paper of 1740 [**6**], Euler obtained one of his finest triumphs, namely,

$$s_{2n}=\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2n}}=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},$$

where the B_{2n} are the Bernoulli numbers (see below). The connection with the Bernoulli numbers was actually established a little later in his *Institutiones calculi differentialis* of 1755 [**8**]. In the 1740 paper he also determined the sum $\Sigma_{\nu=1}^{\infty}(-1)^{\nu-1}(1/\nu^n)$ for the first few odd values of n.

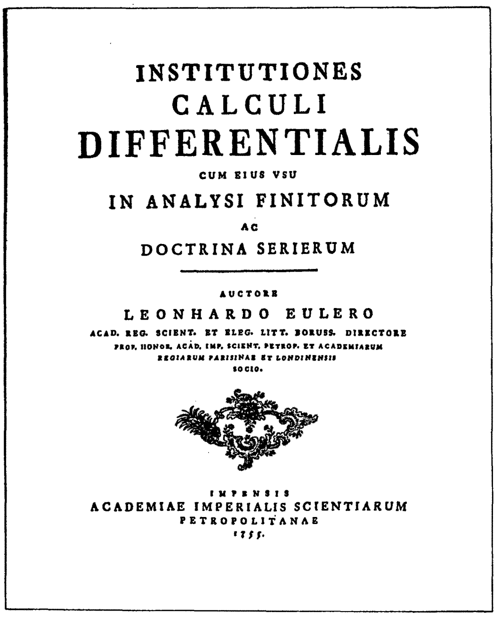
In *Ars conjectandi* (1713), James Bernoulli, who was treating the subject of probability, had introduced the now widely used Bernoulli numbers. Bernoulli had given the following formula for the sum of powers of consecutive positive integers without demonstration:

$$\sum_{k=1}^{n}k^c=\frac{1}{c+1}n^{c+1}+\frac{1}{2}n^c+\frac{c}{2}B_2n^{c-1}+\frac{c(c-1)(c-2)}{2\cdot 3\cdot 4}B_4n^{c-3}$$
$$+\frac{c(c-1)(c-2)(c-3)(c-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}B_6n^{c-5}+\cdots. \tag{10}$$

This series terminates at the last positive power of n, and the B's are the **Bernoulli numbers**:

$$B_2=1/6, B_4=-1/30, B_6=1/42, B_8=-1/30, B_{10}=5/66,\ldots.$$

Bernoulli also gave the recurrence relation which permits one to calculate these coefficients.



INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS VSU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

Another famous result of Euler's, the Euler-Maclaurin summation formula, is a generalization of Bernoulli's formula (10). Let $f(x)$ be a real-valued function of the real variable x with $2k+1$ continuous derivatives on the interval $[0, n]$. Then (in modern notation) Euler's formula is

$$\sum_{i=0}^{n} f(i) = \int_0^n f(x)\,dx + \frac{1}{2}[f(n)+f(0)] + \frac{B_2}{2!}[f'(n)-f'(0)]$$
$$+\frac{B_4}{4!}[f'''(n)-f'''(0)] + \cdots \tag{11}$$
$$+\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n)-f^{(2k-1)}(0)\right]+R_k,$$

where

$$R_k = \int_0^n f^{(2k+1)}(x) P_{2k+1}(x)\,dx.$$

Here n and k are positive integers, and $P_{2k+1}(x)$ is the $(2k+1)$th Bernoulli polynomial (which also appears in Bernoulli's *Ars conjectandi*). It can be represented for $0 \leqslant x \leqslant 1$ by

$$P_{2k+1}(x) = 2(-1)^{k+1} \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{(2m\pi)^{2k+1}}.$$

The Bernoulli numbers B_i are related to the Bernoulli polynomials by

$$P_k(x) = \frac{x^k}{k!} + \frac{B_1 x^{k-1}}{1!(k-1)!} + \frac{B_2 x^{k-2}}{2!(k-2)!} + \cdots + \frac{B_k}{k!},$$

where $B_1 = -\frac{1}{2}$, and $B_{2k+1} = 0$ for $k = 1, 2, \cdots$. They are often defined today by a relation given later by Euler, namely,

$$t(e^t - 1)^{-1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}.$$

Euler's derivation of formula (11) is interesting in its use of the infinitesimal calculus in treating finite series. He begins by noting that if $s(n) = \Sigma_{i=0}^{n} f(i)$, then

$$f(n) = s(n) - s(n-1) = -[s(n-1) - s(n)] = \frac{ds}{dn} - \frac{1}{2!}\frac{d^2s}{dn^2} + \frac{1}{3!}\frac{d^3s}{dn^3} - \cdots; \tag{12}$$

hence (solving for ds/dn and integrating),

$$s = \int f\,dn + \frac{1}{2!}\frac{ds}{dn} - \frac{1}{3!}\frac{d^2s}{dn^2} + \cdots. \tag{13}$$

In order to express the sum s in terms of f, recursion is used. Differentiating (12) repeatedly gives

$$\begin{aligned} \frac{df}{dn} &= \frac{d^2s}{dn^2} - \frac{1}{2!}\frac{d^3s}{dn^3} + \frac{1}{3!}\frac{d^4s}{dn^4} - \cdots, \quad \text{so } \frac{d^2s}{dn^2} = \frac{df}{dn} + \frac{1}{2!}\frac{d^3s}{dn^3} - \cdots. \\ \frac{d^2f}{dn^2} &= \frac{d^3s}{dn^3} - \frac{1}{2!}\frac{d^4s}{dn^4} + \frac{1}{3!}\frac{d^5s}{dn^5} - \cdots, \quad \text{so } \frac{d^3s}{dn^3} = \frac{d^2f}{dn^2} + \frac{1}{2!}\frac{d^4s}{dn^4} - \cdots \end{aligned} \tag{14}$$

and so on. Substituting these values for ds/dn, ds^2/dn^2, $ds^3/dn^3, \cdots$ in (12) gives

$$s = \int f\,dn + \frac{1}{2!}\left[f + \frac{1}{2!}\left(\frac{df}{dn} + \frac{1}{2!}\left(\frac{df^2}{dn^2} + \cdots\right)\right) - \frac{1}{3!}\left(\frac{d^2f}{dn^2}\cdots\right)\right]$$
$$-\frac{1}{3!}\left(\frac{df}{dn} + \frac{1}{2!}\left(\frac{d^2f}{dn^2} + \cdots\right) + \cdots\right) + \frac{1}{4!}\left(\frac{d^2f}{dn^2} - \cdots\right) - \cdots$$

which is cumbersome, but does show the form in which s can be expressed. Euler wrote

$$s=\int f\,dn+\alpha f+\frac{\beta\,df}{dn}+\frac{\gamma\,d^2f}{dn^2}+\frac{\delta\,d^3f}{dn^3}+\cdots$$

and substituted s and its derivatives into (13) to obtain recursion relations for the coefficients of $f, df/dn, d^2f/dn^2$, etc., finally obtaining formula (11). A discussion of Euler's derivation of the Euler-Maclaurin formula as well as some of his interesting applications of it is contained in [**9**, 3.3, 3.4]; a modern summary of Euler's work on the formula is contained in [**1**, p. 1074].

If n is allowed to go to infinity in the Euler-Maclaurin formula (11), the infinite series is divergent for almost all $f(x)$ which occur in applications. Nevertheless, under modest additional hypotheses, the remainder R_k is less than the first term neglected, and so the series and the integral give useful approximations to each other, depending on which is easier to compute.

Independently of Euler, Maclaurin (*Treatise on Fluxions*, 1742) arrived at the same summation formula but by a method a little surer and closer to that which we use today. The remainder was first added and seriously treated by Poisson.

Euler also introduced in his *Institutiones* of 1755 a transformation of series, still known and used [**12**]. Given a series $\Sigma_{n=0}^{\infty}b_n$, he wrote it as $\Sigma_{n=0}^{\infty}(-1)^n a_n$. Then by a number of formal algebraic steps he showed that

$$\sum_{n=0}^{\infty}(-1)^n a_n=\sum_{n=0}^{\infty}(-1)^n\frac{\Delta^n a_0}{2^{n+1}}, \tag{15}$$

where

$$\Delta^0 a_0=a_0,\ \Delta^1 a_0=a_1-a_0,\ \Delta^n a_0=\Delta^{n-1}a_1-\Delta^{n-1}a_0=\Sigma_{i=0}^{n}(-1)^{n-i}\binom{n}{i}a_i,\ n\geqslant 2.$$

His derivation of (15) is as follows. Let $a_n=(-1)^n b_n$ and introduce variables x and y related by $x=y/(1-y)=y+y^2+y^3+\cdots$. Then

$$\begin{aligned}
&b_0x+b_1x^2+b_2x^3+\cdots\\
&=a_0x-a_1x^2+a_2x^3-a_3x^4+\cdots\\
&=a_0(y+y^2+y^3+\cdots)-a_1(y^2+2y^3+3y^4+4y^5+\cdots)\\
&\quad+a_2(y^3+3y^4+6y^5+10y^6+\cdots)-a_3(y^4+4y^5+10y^6+20y^7+\cdots)+\cdots\\
&=a_0y-(a_1-a_0)y^2+(a_2-2a_1+a_0)y^3-\cdots.
\end{aligned}$$

Setting $x=1$ and $y=\frac{1}{2}$ yields

$$\sum_{n=0}^{\infty}b_n=a_0-a_1+a_2-a_3+\cdots=\frac{a_0}{2}-\frac{\Delta a_0}{4}+\frac{\Delta^2 a_0}{8}-\cdots,$$

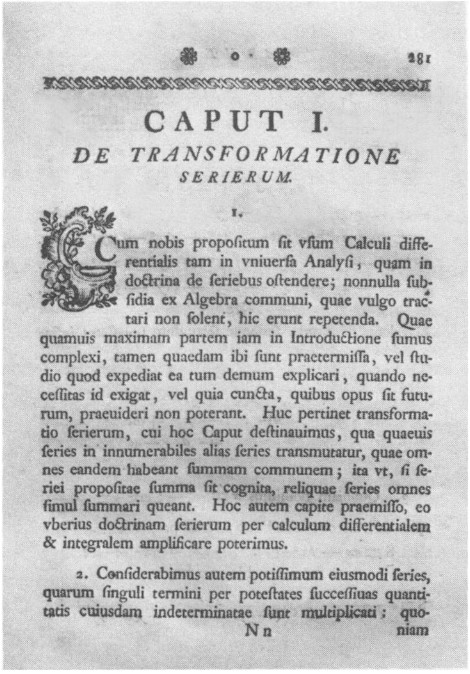
as required.

The transformation in (15) often converts a convergent series into a more rapidly converging one. However, for Euler, who did not usually distinguish between convergent and divergent series, the transformation could also transform divergent series into convergent ones. For, if one applies (15) to the series (1), which is $\Sigma_{n=0}^{\infty}(-1)^n$, then since $a_0=1$ and $\Delta^n a_0=0$ for all $n>1$, the sum on the right is $1/2$. Likewise for the series

$$1-2+2^2-2^3+2^4\cdots,$$

the transformation in (15) gives

$$\sum_{n=0}^{\infty}(-1)^n2^n=\frac{1}{2}(1)+\frac{1}{4}(-1)+\frac{1}{8}(1)-\frac{1}{16}(-1)+\cdots=\frac{1}{3}.$$



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CAPUT I.

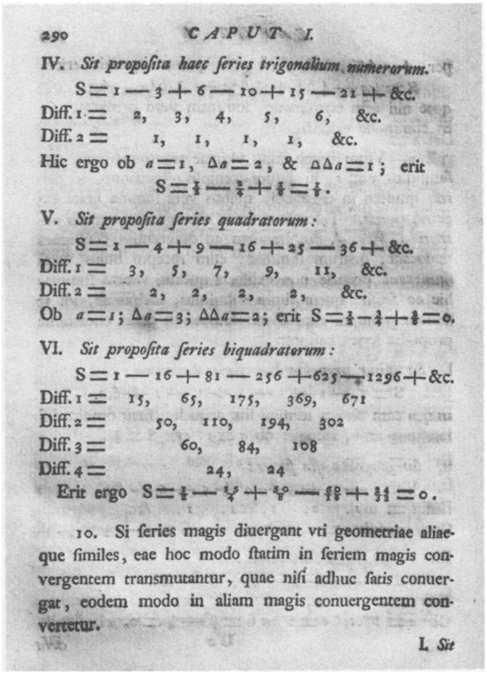
DE TRANSFORMATIONE SERIERUM.

1.

Cum nobis propoſitum ſit vſum Calculi differentialis tam in vniuerſa Analyſi, quam in doctrina de ſeriebus oſtendere; nonnulla ſubſidia ex Algebra communi, quae vulgo tractari non ſolent, hic erunt repetenda. Quae quamuis maximam partem iam in Introductione ſumus complexi, tamen quaedam ibi ſunt praetermiſſa, vel ſtudio quod expediat ea tum demum explicari, quando neceſſitas id exigat, vel quia cuncta, quibus opus ſit futurum, praeuideri non poterant. Huc pertinet transformatio ſerierum, cui hoc Caput deſtinauimus, qua quaeuis ſeries in innumerabiles alias ſeries transmutatur, quae omnes eandem habeant ſummam communem; ita vt, ſi ſeriei propoſitae ſumma ſit cognita, reliquae ſeries omnes ſimul ſummari queant. Hoc autem capite praemiſſo, eo vberius doctrinam ſerierum per calculum differentialem & integralem amplificare poterimus.

2. Conſiderabimus autem potiſſimum eiusmodi ſeries, quarum ſinguli termini per poteſtates ſucceſſiuas quantitatis cuiusdam indeterminatae ſunt multiplicati; quo-

Nn niam



290 CAPUT I.

IV. *Sit propoſita haec ſeries trigonalium numerorum.*

$$S = 1 - 3 + 6 - 10 + 15 - 21 + \&c.$$

Diff. 1 = 2, 3, 4, 5, 6, &c.

Diff. 2 = 1, 1, 1, 1, &c.

Hic ergo ob $a = 1$, $\Delta a = 2$, & $\Delta\Delta a = 1$; erit

$$S = \tfrac{1}{2} - \tfrac{2}{4} + \tfrac{1}{8} = \tfrac{1}{8}.$$

V. *Sit propoſita ſeries quadratorum:*

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \&c.$$

Diff. 1 = 3, 5, 7, 9, 11, &c.

Diff. 2 = 2, 2, 2, 2, &c.

Ob $a = 1$; $\Delta a = 3$; $\Delta\Delta a = 2$; erit $S = \tfrac{1}{2} - \tfrac{3}{4} + \tfrac{2}{8} = 0.$

VI. *Sit propoſita ſeries biquadratorum:*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \&c.$$

Diff. 1 = 15, 65, 175, 369, 671

Diff. 2 = 50, 110, 194, 302

Diff. 3 = 60, 84, 108

Diff. 4 = 24, 24

Erit ergo $S = \tfrac{1}{2} - \tfrac{15}{4} + \tfrac{50}{8} - \tfrac{60}{16} + \tfrac{24}{32} = 0.$

10. Si ſeries magis diuergant vti geometriae aliaeque ſimiles, eae hoc modo ſtatim in ſeriem magis convergentem transmutantur, quae niſi adhuc ſatis conuergat, eodem modo in aliam magis conuergentem convertetur.

I. Sit

Euler demonstrates (*Institutiones*, Pars Posterior, Chap. I) his transformation of series with many examples. Here he "sums" the alternating series of triangular numbers, $\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)/2 = 1/8$, the alternating series of squares, $\sum_{n=0}^{\infty}(-1)^n(n+1)^2 = 0$ and of fourth powers $\sum_{n=0}^{\infty}(-1)^n(n+1)^4 = 0$.

These results are the same as those Euler got by taking the sum of the series to be the value of the function from which the series is derived.

Euler took up the subject of sums of series in a major paper of 1754/55 entitled "On Divergent Series" [5], in which he recognized the distinction between convergent and divergent series. Apropos of the former he says that for those series in which by constantly adding terms we approach closer and closer to a fixed number, which happens when the terms continually decrease, the series is said to be convergent and the fixed number is its sum. Series whose terms do not decrease and may even increase are divergent.

On divergent series, Euler says one should not use the term "sum" because this refers to actual addition. Euler then states a general principle which explains what he means by the definite value of a divergent series. He points out that the divergent series comes from finite algebraic expressions and then says that the value of the series is *the value of the algebraic expression from which the series comes*. Euler further states, "Whenever an infinite series is obtained as the development of some closed expression, it may be used in mathematical operations as the equivalent of that expression, even for values of the variable for which the series diverges." He repeats this principle in his *Institutiones* of 1755:

> Let us say, therefore, that the sum of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense the sum of the infinite series $1 - x + x^2 - x^3 + \cdots$ will be $1/(1+x)$, because the series arises from the expansion of the fraction, whatever number is put in place of x. If this is agreed, the new definition of the word sum coincides with the ordinary meaning when a series converges; and since divergent series have no sum in the proper sense of the word, no inconvenience can arise from this terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections.

It is fairly certain that Euler meant to limit this doctrine to power series.

Other 18th-century mathematicians also recognized that a distinction must be made between what we now call convergent and divergent series, though they were not at all clear as to what the distinction should be. They were dealing with a new concept and, like all pioneers, they had to struggle to clear the forest. Certainly the interpretation of series suggested by Newton, and adopted by Leibniz, Euler, and Lagrange, that series are just long polynomials and so belong in the domain of algebra, could not serve as a rigorous foundation for the work with series.

One outstanding characteristic of the 18th-century investigations is that mathematicians trusted the symbols far more than logic. Because infinite series had the same symbolic form for all values of x, the distinction between values of x for which the series converged and values for which they diverged did not seem to demand much attention. And even though it was recognized that some series, such as $1 + 2 + 3 + \cdots$, had infinite sums, mathematicians preferred to try to give meaning to the sums rather than question the applicability of summation. Of course, they were fully aware of the need for some proofs. We have seen that Euler did try to justify his use of divergent series. But the few efforts to achieve rigor, significant because they show that standards of rigor vary with the times [**10**], did not validate the work of the century, and mathematicians almost willingly took the position that what cannot be cured must be endured.

Though we have only glimpsed some of Euler's work, almost all of the great mathematicians of the 18th century contributed to the subject of infinite series [**13**]. It is fair to say that in this work the formal view dominated. Aware of the power of formal manipulation, mathematicians either ignored or deferred consideration of any limitations to their techniques, such as the importance of convergence. Their work produced useful results, and they were satisfied with this pragmatic sanction. They exceeded the bounds of what they could justify, but they were at least prudent in their use of divergent series. However, these 18th-century mathematicians were to have the last word. Dimly, they saw in divergent infinite series, ideas which were later to gain acceptance, namely, summability and asymptotic series [**2**], [**11**], [**13**, chapter 47], [**14**].

I wish to thank Professor Edward J. Barbeau of the University of Toronto for his critique and for supplying some material on Euler's proofs.

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Glossary

Leonhard Euler's vast contribution to mathematics can be glimpsed in the many terms, formulas, equations, and theorems which today bear his name. An exhaustive list of such terms would be difficult to compile; even harder would be a list all of whose entries could be carefully verified as having originated in Euler's work. This compilation contains those items which can be readily found in mathematics texts and reference works; although impressive, certainly the list is incomplete.

The glossary was produced through the efforts of Karl Anderson and Jeff Ondich, students at St. Olaf College, with the assistance of Lynn Steen, Gerald Alexanderson, and members of the editorial board of *Mathematics Magazine*. Definitions of several of the entries vary according to the source consulted; we have chosen descriptions that seem most common. Each entry in the glossary can be found in one or more of the reference works or in one of the texts listed in the **References**.

Although the symbols do not bear his name, Euler introduced many of the modern conventions of mathematical notation—most notably, the symbol $f(x)$ for a function, the notations $\sin x$, $\cos x$ for sine and cosine functions, the symbols Σ for summation, Δy, $\Delta^2 y$, etc., for finite differences, e for the base of the natural logarithm, and i for $\sqrt{-1}$. In addition, it is easy to point out numerous mathematical terms and theorems missing from our list (because they bear some other mathematician's name), but which are rightfully attributed to Euler. R. A. Raimi has noted, "There is ample precedent for naming laws and theorems for persons other than their discoverers, else half of analysis would be named for Euler" (*Amer. Math. Monthly*, 83 (1976) 522).

Terms

Euler Angles. These three angles are commonly used to fix the directions of a new set of rectangular space coordinates x^*, y^*, z^* with reference to an old set x, y, z. They are usually taken as the angle between the z^* and z-axes, the angle between the x-axis and the line l of intersection of the x^*y^* and xy-planes (l is called the nodal line), and the angle between the x^*-axis and l. (See FIGURE 1; see Euler's theorem for rotation of coordinate system.)

The Euler Characteristic. For a polyhedral surface, this is the number $\chi = V - E + F$, where V is the number of vertices, E the number of edges, and F the number of faces. More generally, for an n-dimensional simplicial complex K, the Euler (or Euler-Poincaré) characteristic is defined by

$$\chi = \sum_{i=0}^{n} (-1)^i s(i)$$

where $s(i)$ is the number of i-dimensional simplexes in K.

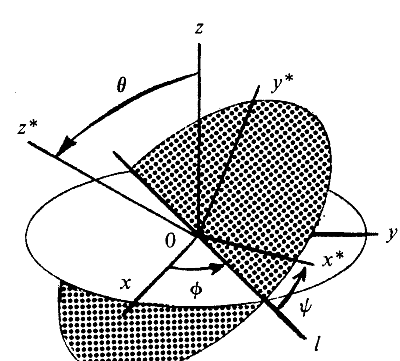


FIGURE 1. The xyz coordinate axes rotate into the $x^*y^*z^*$ coordinate axes by successive rotations through the Euler angles ϕ, θ, ψ.

Copper engraving of Leonhard Euler by S. G. Kütner, 1780, based on the portrait by J. Darbes, 1778; Kunstmuseum, Genf.

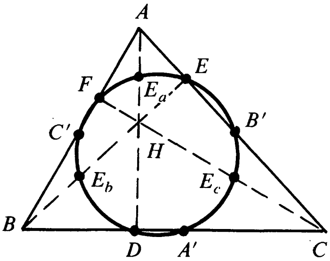


FIGURE 2. **The midpoints of the sides of $\triangle ABC$ are A', B', C', the feet of its altitudes are D, E, F, and the point of their intersection, the orthocenter, is H. The Euler points, E_a, E_b, E_c, are the midpoints of segments AH, BH, and CH, respectively.**

Euler's (Nine-point) Circle. This circle passes through the midpoints of the sides of a triangle, the feet of its altitudes, and the midpoints of the line segments between its vertices and its orthocenter; these last 3 points are called the **Euler points** of the triangle. See FIGURE 2. Euler proved that *for any triangle, the feet of its altitudes and the midpoints of its sides all lie on one circle*, but it was Poncelet (1788–1867) who showed that the Euler points also lie on this same circle. Poncelet named this circle the nine-point circle; it is also called Feuerbach's circle.

Euler Circuit. This is a closed path of edges in a graph in which each edge appears exactly once. Euler's theorem for graphs states that *an undirected graph has an Euler circuit if and only if it is connected and all of its vertices have even degree* (the degree of a vertex is the number of edges which meet at that vertex). An 18th century Sunday pastime—to stroll a continuous path, attempting to cross each of the seven bridges of Königsberg exactly once—was the source of the theorem.

Euler's Constant γ. The constant $\gamma = .577215665\ldots$, which was calculated by Euler to 16 decimal places, is defined by the limit

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right).$$

Sometimes called the Euler-Mascheroni constant, it can also be defined by the integral

$$\gamma = \int_{\infty}^{0} e^{-t} \ln t \, dt.$$

It is not known if γ is an irrational number.

Euler (-Venn) Diagram. Such a diagram consists of closed curves, used to represent relations between logical propositions or sets. See FIGURE 3.

Euler Line. The line defined by the centroid (the intersection of the medians), the orthocenter (the intersection of the altitudes), and the circumcenter (the intersection of the perpendicular bisectors) of a triangle. In addition to the remarkable collinearity of these three points, the distance between the centroid and the circumcenter is equal to half the distance between the centroid and the orthocenter. See FIGURE 4.

Euler Numbers. The Euler numbers E_n can be defined by the infinite series

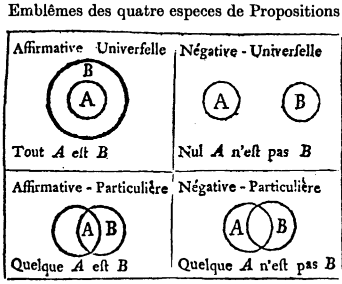
$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} z^{2n};$$

alternately, they can be defined using the Euler polynomial $E_n(x)$ (see **Functions** below):

$$E_n = 2^n E_n(1/2).$$

Some properties are: $E_0 = 1$, $E_{2k+1} = 0$, all k, $E_2 = -1$, $E_4 = 5$, $E_6 = 61$;

$$\sum_{j=0}^{n} \binom{2n}{2j} E_{2j} = 0, \text{ all } n \geqslant 1.$$



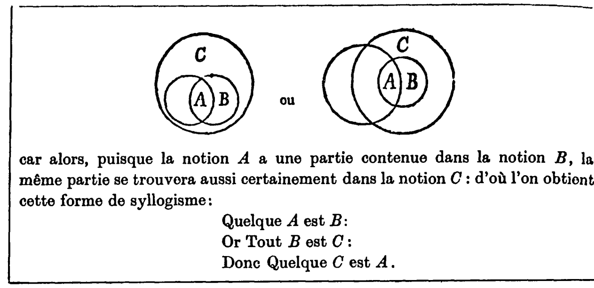


FIGURE 3. **Euler diagrams from letters 103, 104 of *Lettres*.... Reproduced from *Opera Omnia* (3) 11, in which the diagrams are facsimile reproductions from the first edition.**

Eulerian Numbers. The numbers $A_{n,k}$ were defined by Euler as follows: If $H_n(\lambda)$ is the rational function of λ defined by the generating function

$$\frac{1-\lambda}{e^t-\lambda}=\sum_{n=0}^{\infty} H_n(\lambda)\frac{t^n}{n!}, \lambda \neq 1,$$

then the Eulerian numbers $A_{n,k}$ are defined by the polynomial

$$(\lambda-1)^n H_n(\lambda)=\sum_{k=1}^{n} A_{n,k}\lambda^{k-1}.$$

(The polynomials $A_n(\lambda)=\sum_{k=1}^{n} A_{n,k}\lambda^k$ are called **Eulerian polynomials**.) The numbers $A_{n,k}$ have important combinatorial properties; a combinatorial formula due to Euler is

$$A_{n,k}=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n, k=0,1,\ldots,n,$$

and another (Worpetzky, 1883) is

$$x^n=\sum_{k=1}^{n} A_{n,k}\binom{x+k-1}{n}.$$

Today, the number $A_{n,k}$ is commonly defined as the number of permutations of the set $\{1,2,\ldots,n\}$ having $k-1$ descents (a descent of a permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$$

is a pair a_i, a_{i+1} with $a_i > a_{i+1}$). From this, the properties $A_{n,k}=A_{n,n-k+1}$ $(1 \leqslant k \leqslant n)$ and $\sum_{k=1}^{n} A_{n,k}=n!$ are evident.

Euler Spherical Triangles. If three points A, B, C are on a sphere such that no pair of these are diametrically opposite, then they determine three great circles, each of which joins a pair of the points. These three great circles divide the sphere into eight spherical triangles, whose sides are arcs of great circles, and have length less than π; these are called Euler spherical triangles. See FIGURE 5.

Euler Triangle. Given any triangle, its Euler triangle has as vertices the three Euler points of the triangle (see FIGURE 2).

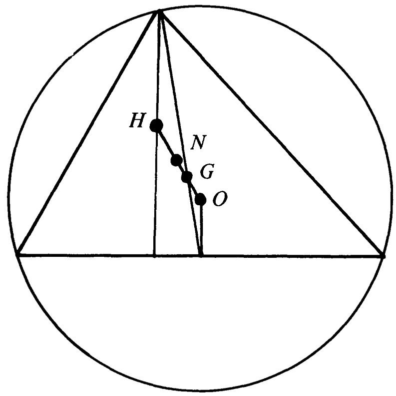


FIGURE 4. **The orthocenter *H*, centroid *G* and circumcenter *O* determine the Euler line of a triangle. The center *N* of the nine-point circle is also on this line, and $3OG = OH$, $2ON = OH$.**

Functions

Euler's First Integral (Beta function). The beta function is defined by the integral

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt,\quad R(z)>0, R(w)>0.$$

It is related to Euler's second integral, the gamma function, by the equation

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}=B(w,z).$$

Euler's Second Integral (Gamma function). Euler's integral defines the gamma function as

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt, R(z)>0.$$

This function satisfies the equation $\Gamma(z+1)=z\Gamma(z)$ for all $R(z)>0$, hence $\Gamma(n+1)=n!$ for all positive integers n. Other formulas for $\Gamma(z)$ also due to Euler are

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)},\quad z\neq 0,-1,-2,\ldots$$

and

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}, z\neq 0,-1,-2,\ldots.$$

Euler Polynomials. The polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad |t|<\pi.$$

(The Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!};$$

there are many equations relating the $E_n(x)$ and $B_n(x)$.) Some properties of the Euler polynomials are:

$$E_n'(x)=nE_{n-1}(x)$$

$$E_n(x+1)+E_n(x)=2x^n$$

$$\sum_{k=1}^{m}(-1)^{m-k}k^n=\frac{1}{2}\left(E_n(m+1)+(-1)^m E_n(0)\right).$$

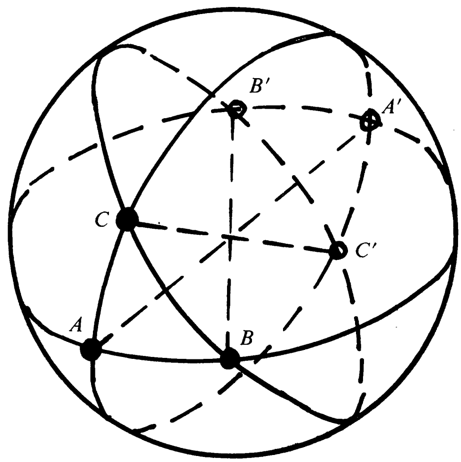


FIGURE 5. **Euler spherical triangles.**

Euler Product. Under certain conditions, the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\quad (s \text{ real or complex})$$

has a representation as a formal product

$$F(s)=\prod_{p \text{ prime}} F_p(s),$$

called an Euler product; the functions

$$F_p(s)=1+f(p)p^{-s}+f(p^2)p^{-2s}+f(p^3)p^{-3s}+\cdots$$

are called **Euler factors**. The premier example is the zeta function, for which $f(n)$ is the constant function 1 (see Euler's identity, below). Another example, derived from this, is

$$\begin{aligned}\frac{\zeta(2s)}{\zeta(s)}&=\prod_p\left(\frac{1-p^{-s}}{1-p^{-2s}}\right)=\prod_p\left(1+\frac{1}{p^s}\right)^{-1}=\prod_p\left(1-p^{-s}+p^{-2s}-p^{-3s}+\cdots\right)\\&=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s},\end{aligned}$$

where $\lambda(n)=(-1)^{\rho}$, ρ defined as $\sum_{i=1}^{k}\alpha_i$ if $n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ is the prime factorization of n.

Euler's ϕ function (Euler's totient function). For each positive integer n, $\phi(n)$ is defined as the number of positive integers less than n and relatively prime to n. If $p_1, p_2, \ldots, p_k$ are the distinct prime factors of n, then

$$\phi(n)=n\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right).$$

Euler's Identity (Zeta function). For $R(s)>1$,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},$$

the product taken over all primes. The sum on the left is known today as the Riemann zeta function.

Formulas

Euler-Binet formula. The rational solutions to the equation

$$w^3 + 3w(x^2 + y^2 + z^2) + 6xyz = 0$$

are given by

$$w = -6pabc, \quad x = pa(a^2 + 3b^2 + 3c^2),$$
$$y = pb(a^2 + 3b^2 + 9c^2), \quad z = 3pc(a^2 + b^2 + 3c^2).$$

Here, $(a, b, c) = 1$ and p is rational. The Euler-Binet formula is used to find solutions to the equation

$$x^3 + y^3 + z^3 + w^3 = 0.$$

Euler Force (critical load) for a beam or column. The Euler force is the maximum axial load that a long, slender beam or column can carry without buckling. This critical force is given by the formula

$$K\pi^2 \frac{YI}{L^2},$$

where I is the beam's moment of inertia of cross-sectional area, L is its unsupported length, Y is its stiffness, and K is a constant that depends on the conditions of end support of the beam ($K = 1$ if both ends are free to turn).

Euler's formula for $e^{i\theta}$. This fundamental rule links trigonometric to exponential functions:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

When $\theta = \pi$ and 2π, Euler's formula yields the famous, surprising results

$$e^{i\pi} = -1 \text{ and } e^{i2\pi} = 1.$$

Euler-Fourier formulas. In the Fourier series expansion of the function $F(x)$,

$$F(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \; -\pi < x < \pi,$$

the Euler-Fourier coefficients are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\, dx, \; b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\, dx.$$

Euler-Maclaurin sum formula. This summation formula is one of the most important in the calculus of finite differences; it was discovered by Euler and independently by Maclaurin in the decade 1732–42. One version of the formula is: If $f(x)$ has its first $2n$ derivatives continuous on an interval $[0, m]$, m an integer, then

$$\sum_{k=0}^{m} f(k) = \int_0^m f(x)\, dx + \frac{1}{2}(f(0) + f(m))$$
$$+ \sum_{k=1}^{n} \frac{B_{2k}}{(2k!)}(f^{(2k-1)}(m) - f^{(2k-1)}(0)) + R_n,$$

where the B_{2n} are the Bernoulli numbers (see: Euler numbers), and the remainder term R_n may be given in several forms. One form is

$$R_n = \frac{B_{2n}}{(2n)!}\sum_{k=0}^{m-1} f^{(2n)}(k+\theta), \text{ for some } 0 < \theta < 1;$$

another is given by M. Kline, this *Magazine*, p. 311.

In its simplest form, the summation formula for a function $f(x)$ with a continuous derivative on $[0, m]$ is

$$\sum_{k=0}^{m} f(k) = \int_0^m f(x)\,dx + \frac{1}{2}(f(0) + f(m)) + \int_0^m \left(x - [x] - \frac{1}{2}\right) f'(x)\,dx,$$

where $[x]$ is the greatest integer function.

Equations

Euler's equation in the calculus of variations. Problem: Find a curve $y(x)$ joining the points (x_1, y_1) and (x_2, y_2) such that the integral

$$\int_{x_1}^{x_2} I(x, y, y')\,dx$$

is a maximum or minimum (the integral has a stationary value). A necessary condition for $y(x)$ to be a solution to the problem is that y satisfy the Euler (-Lagrange) equation:

$$\frac{\partial I}{\partial y} - \frac{d}{dx}\frac{\partial I}{\partial y'} = 0, \text{ where } y' = \frac{dy}{dx}.$$

Euler's (equidimensional) equation. Also called the Euler-Cauchy equation, this is an nth order differential equation of the form

$$z^n w^{(n)} + b_1 z^{n-1} w^{(n-1)} + \cdots + b_n w = 0,$$

where $w(z)$ is a function of z, and the b_i are constants. The substitution $z = e^s$ transforms this into an equation with constant coefficients, i.e., if $\tilde{w}(s) = w(e^s)$, then the equation becomes

$$\tilde{w}^{(n)} + c_1 \tilde{w}^{(n-1)} + \cdots + c_n \tilde{w} = 0,$$

the c_i constants. The second-order equation is most often cited, i.e.,

$$x^2 y'' + pxy' + qy = 0.$$

Euler's equation of motion for an ideal compressible or incompressible fluid. The Eulerian method of analyzing fluid flow focuses on each position in space and observes how the fluid motion varies over time at that position. Euler's equation is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p,$$

where $\mathbf{v}$ is the velocity field, p is the pressure, ρ is the density, and $\mathbf{F}$ is the external force per unit of mass of the fluid.

Euler's equations (of motion) for the rotation of a rigid body. The basic equations for the rotation of a rigid body are

$$N_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_3\omega_2$$

$$N_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_1\omega_3$$

$$N_1 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_2\omega_1$$

where the N_i are torques, the I_i are the principal moments of inertia, and the ω_i are angular velocities about the three coordinate axes.

Euler's equation on normal curvature. If κ_1 and κ_2 are the principal normal curvatures at point P on a surface S, and κ is the normal curvature in the direction making an angle θ with the direction having normal curvature κ_1, then

$$\kappa=\kappa_1\cos^2\theta+\kappa_2\sin^2\theta.$$

Techniques

Euler multiplier method of solving a differential equation. If a differential equation $y'g(x, y)+h(x, y)=0$ is not exact, the equation is multiplied by a function $\mu(x, y)$, called an Euler multiplier, or integrating factor, so that the product $y'g\mu+h\mu$ is a perfect derivative. A partial differential equation for the determination of $\mu(x, y)$ is

$$\frac{\partial(g\mu)}{\partial x}=\frac{\partial(h\mu)}{\partial y}.$$

Euler's numerical method for the solution of differential equations. This iterative scheme is used to find a numerical solution of an ordinary differential equation, $y'=f(x, y)$. The recursion formula is

$$y_{n+1}=y_n+hf(x_n, y_n)$$

where $h=x_{n+1}-x_n$ is the step size between successive points. The error in this method is often substantial, and the **improved Euler method** is used, which is based on approximation by the trapezoidal rule. The recursion formula is

$$y_{n+1}=y_n+\frac{h}{2}\left[f(x_n, y_n)+f(x_{n+1}, z_{n+1})\right],$$

where $z_{n+1}=y_n+hf(x_n, y_n)$.

Euler's transformation of a series. The series $\Sigma_{k=0}^{\infty}(-1)^k a_k$ is transformed into the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+1}}\Delta^k a_0,\text{ where }\Delta^0 a_0=a_0,\text{ and }\Delta^k a_0=\sum_{m=0}^{k}(-1)^m\binom{k}{m}a_{k-m}, k\geqslant 1.$$

If the original series converges, the transformed series converges (often more quickly) to the same sum.

Theorems

Euler's Addition Theorem for elliptic integrals. *If* $g(x)=(1-x^2)(1-k^2x^2)$, *then*

$$\int_0^a\frac{dx}{\sqrt{g(x)}}+\int_0^b\frac{dx}{\sqrt{g(x)}}=\int_0^c\frac{dx}{\sqrt{g(x)}}$$

where

$$c=\frac{b\sqrt{g(a)}+a\sqrt{g(b)}}{\sqrt{1-k^2a^2b^2}}.$$

Euler's Criterion for quadratic residues. An integer a is a quadratic residue of an integer b if there is a solution to the congruence $x^2\equiv a \bmod b$. Euler's criterion says that *a is a quadratic residue of the odd prime p if and only if* $a^{(p-1)/2}\equiv 1 \bmod p$.

Euler's generalization of Fermat's Theorem. *If n and a are positive integers which are relatively prime, then*

$$a^{\phi(n)}\equiv 1 \bmod n.$$

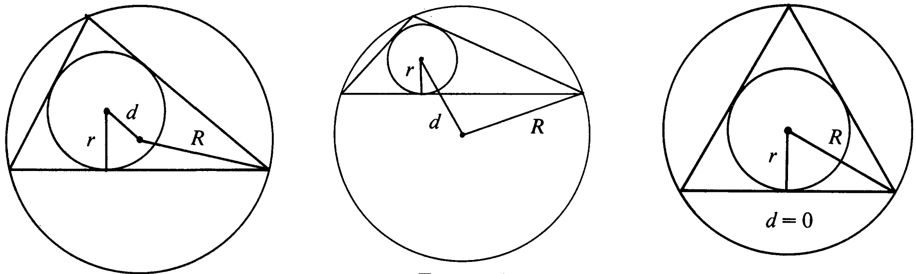


FIGURE 6

Euler's theorem for homogeneous functions. *If $f\colon R^n \to R$ is a function that satisfies*

$$f(\lambda \mathbf{x}) = \lambda^k f(\mathbf{x})$$

for all $\mathbf{x} \in R^n$ and $\lambda \in R$, then

$$x \cdot \nabla f(\mathbf{x}) = kf(\mathbf{x}).$$

Euler-Lagrange Theorem. *Any positive integer can be expressed as the sum of at most four squares.*

Euler's Officer Problem. Euler's analysis of the 36 officer problem—the assignment of six different officers from each of six regiments to six squads, each including an officer of every rank and a member of every regiment—led him to conjecture that there exist no such pairs of orthogonal Latin squares of order $n \equiv 2 \bmod 4$. Although Euler was right for $n = 6$, his conjecture has been proved false for all $n > 6$.

Euler's theorems on partitions. Many theorems in the theory of partitions were first proved by Euler, and bear his name; we give two of these. *The number of partitions of a positive integer n into odd parts is equal to the number of partitions of n into distinct parts.* For example, $n = 6$ is partitioned into odd parts in four ways: $5+1$, $3+3$, $3+1+1+1$, $1+1+1+1+1+1$, and into distinct parts in four ways: 6, $5+1$, $4+2$, $3+2+1$.

Let $p(k)$ denote the number of partitions of k for $k \geqslant 1$, and define $p(k) = 0$ for k negative, and $p(0) = 1$. *If $n > 0$, then*

$$\begin{aligned} p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \cdots \\ + (-1)^m p\left(n - \frac{m}{2}(3m-1)\right) + (-1)^m p\left(n - \frac{m}{2}(3m+1)\right) + \cdots = 0. \end{aligned}$$

Euler's theorem for pentagonal numbers. *If $|x| < 1$, then*

$$\begin{aligned} \prod_{m=1}^{\infty} (1 - x^m) &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots \\ &= 1 + \sum_{n=-\infty}^{\infty} (-1)^n x^{\omega(n)} \end{aligned}$$

where $\omega(n) = \dfrac{3n^2 - n}{2}$ is the nth pentagonal number.

Euler's theorem for points on a line. *If the points P, Q, R and S are collinear, then*

$$PQ \cdot RS + PR \cdot SQ + PS \cdot QR = 0.$$

Euler's theorem for polyhedra. *For any convex polyhedron, the Euler characteristic is equal to 2; that is, $V - E + F = 2$.*

Euler's theorem for primes. *The sum $\Sigma(1/p)$, and the product $\Pi(1 - (1/p))^{-1}$ are both divergent, as p runs through all the primes. It follows that there are an infinite number of primes.*

Euler's theorem for rotation of coordinate system (or rigid body). *If two rectangular coordinate systems with the same origin and arbitrary directions of axes are given in 3-space, there exists a line through the origin such that one coordinate system is transformed into the other by a rotation about this line.* (The transformation can also be achieved by three successive rotations through the Euler angles ϕ, θ and ψ; see FIGURE 1.)

Euler's recursion theorem for the sums of divisors. *Let $\sigma(k)$ be the sum of the divisors of the positive integer k, and define $\sigma(k)=0$ for $k \leqslant 0$. Define the sum $S(n)$:*

$$S(n)=\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7) -\sigma(n-12)-\sigma(n-15)+\cdots+(-1)^m\sigma\left(n-\frac{m}{2}(3m-1)\right) +(-1)^m\sigma\left(n-\frac{m}{2}(3m+1)\right)+\cdots.$$

Then $S(n)=(-1)^{m-1}n$ if $n=\frac{m}{2}(3m\pm 1)$ for some m, and $S(n)=0$ otherwise.

Euler's theorem for a triangle. *The distance d between the circumcenter and incenter of a triangle is given by the equation $d^2=R(R-2r)$, where R, r are the circumradius and inradius, respectively.* See FIGURE 6.

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PROBLEMS

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Proposals

To be considered for publication, solutions should be mailed before April 1, 1984.

1178. Evaluate

$$\lim_{n\to\infty}\left[\prod_{i=1}^{n}\left(a+\frac{i-1}{n}\right)\right]^{1/n},$$

where a is any positive constant. [*Russell Euler, Northwest Missouri State University.*]

1179. Let A be a square matrix of rank r. Show that the minimum polynomial of A has degree at most $r+1$. [*William P. Wardlaw, U.S. Naval Academy.*]

1180. Let R be an associative ring with no nonzero nilpotent elements (i.e., R is reduced). Show that if $x \in R$ has only finitely many distinct powers, then there exists an integer $n > 1$ such that $x = x^n$. [*Gary F. Birkenmeier, Southeast Missouri State University.*]

1181. Let $A_1A_2A_3$ be a triangle and M an interior point. The straight lines MA_1, MA_2, MA_3 intersect the opposite sides at the points B_1, B_2, B_3, respectively. Show that if the areas of triangles A_2B_1M, A_3B_2M, and A_1B_3M are equal, then M coincides with the centroid of triangle $A_1A_2A_3$. [*George Tsintsifas, Thessaloniki, Greece.*]

Quickie

Solution to the Quickie appears on page 328.

Q687. Show that the partial sums of $\displaystyle\sum_{p \text{ prime}} \frac{1}{p}$ are never integers.

[*Lee Whitt, Yorktown, Virginia.*]

ASSISTANT EDITORS: DANIEL B. SHAPIRO and WILLIAM A. MCWORTER, JR., *The Ohio State University.*

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Answer

Solution to the Quickie which appears on page 326.

Q687. More generally, let $p_1, p_2, \dots, p_n$ be $n \geqslant 1$ distinct primes. Suppose that

$$\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}=k\in\mathbb{Z}.$$

Let $N=p_1p_2\cdots p_{n-1}$ (with $N=1$ if $n=1$). Then

$$\frac{N}{p_1}+\frac{N}{p_2}+\cdots+\frac{N}{p_n}=Nk\in\mathbb{Z}.$$

But $N/p_i\in\mathbb{Z}$ for $1\leqslant i<n$, and $N/p_n\notin\mathbb{Z}$, which is a contradiction.

A similar argument will show that the sum of the reciprocals of distinct positive odd integers, plus the reciprocal of an even integer, cannot be an integer.

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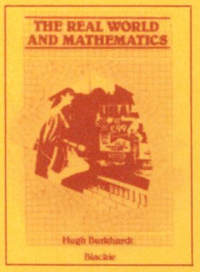

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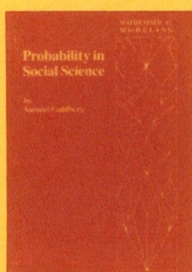

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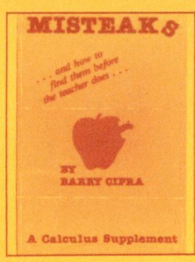

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